EFFICIENT ADAPTIVE ALGORITHMS FOR ELLIPTIC PDEs WITH RANDOM DATA

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Introduction

Developing efficient numerical algorithms for PDEs with random input is critical for reliable uncertainty quantification. Along with more classical Monte Carlo (MC) methods, the **stochastic Galerkin FEM (sGFEM)** has become very popular in the last decade. It produces solutions in tensor product spaces including finite element spaces associated with the spatial domain and spaces of multivariate polynomials for the stochastic discretisation. Adaptive sGFEM algorithms employing a posteriori error estimation should be designed in order to ensure a balanced refinement of both spatial and stochastic components contributing to the total error.

Parametric model problem

Let $D \subset \mathbb{R}^2$ be a bounded spatial domain, $\Gamma_m := \prod_{m\geq 1} \Gamma_m$ be the **parameter domain**, and $\mathbf{y} = (y_1, y_2, \dots) \in \Gamma$ be a multi-variate random parameter. We consider the following parametric steady-state diffusion problem

$$(1)\qquad \begin{aligned} -\nabla\cdot(a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})) &= f(\mathbf{x}), \quad \mathbf{x}\in D,\ \mathbf{y}\in\Gamma,\\ u(\mathbf{x},\mathbf{y}) &= 0, \qquad \mathbf{x}\in\partial D,\ \mathbf{y}\in\Gamma, \end{aligned} \qquad \text{where} \quad a(\mathbf{x},\mathbf{y}) = a_0(\mathbf{x}) + \textstyle\sum_{m\geq 1} a_m(\mathbf{x})\, y_m,\ \forall\,\mathbf{x}\in D,\ \forall\,\mathbf{y}\in\Gamma,$$

is a second-order positive **random field** such that $0 < \alpha_0^{\min} \leq a_0(\mathbf{x}) \leq \alpha_0^{\max}$ and $\sum_{m=1}^{\infty}\|a_m\|_{L^\infty(D)} < \alpha_0^{\min}$. We assume that $\pi = \prod_{m\geq 1}\pi_m(y_m)$ is a product (measure) of Borel measures π_m on $\Gamma_m := [-1,1]$.

Weak formulation. Consider the following Bochner space $V := H_0^1(D)\otimes L^2_\pi(\Gamma)$. We want to find $u\in V$ such that

$$B(u,v) := B_0(u,v) + \textstyle\sum_{m\geq 1} B_m(u,v) = \int_\Gamma \int_D f(\mathbf{x})\, v(\mathbf{x},\mathbf{y})\, d\mathbf{x}d\pi(\mathbf{y}) =: F(v) \quad \forall\, v\in V,$$

where

$$B_0(u,v) := \int_\Gamma \int_D a_0(\mathbf{x})\nabla u\cdot\nabla v\, d\mathbf{x}d\pi(\mathbf{y}) \qquad \text{and} \qquad B_m(u,v) := \int_\Gamma y_m \int_D a_m(\mathbf{x})\nabla u\cdot\nabla v\, d\mathbf{x}d\pi(\mathbf{y}),\ \forall m\geq 1.$$

Stochastic Galerkin discretisation

The sGFEM projects the solution u onto a finite-dimensional subspace $V_{X\mathfrak{P}} := X\otimes\mathcal{P}_{\mathfrak{P}}$ of V. Here, $X = X(\mathcal{T}) \subset H_0^1(D)$ is a **first-order finite element space** associated with a mesh $\mathcal{T}$, while $\mathcal{P}_{\mathfrak{P}} \subset L^2_\pi(\Gamma)$ is a **polynomial space** on Γ defined as

$$\mathcal{P}_{\mathfrak{P}} := \operatorname{span}\{P_\nu(\mathbf{y}) := \textstyle\prod_{m\geq 1} p^m_{\nu_m}(y_m) \ :\ \nu\in\mathfrak{P}\},$$

where $\{p_k^m\}_{k\geq 0}$ is an orthonormal basis of $L^2_{\pi_m}(\Gamma_m)$ and $\mathfrak{P}$ is a finite-dimensional subset of

$$\mathfrak{I} := \Big\{\nu = (\nu_1,\nu_2,\dots)\in\mathbb{N}_0^{\mathbb{N}} \ :\ \#\mathrm{supp}(\nu) < \infty\Big\}.$$

The set $\mathfrak{P}$ is called **index set** and each $\nu\in\mathfrak{P}$ is called **index**. In this way, the discrete weak formulation is: find $u_{X\mathfrak{P}}\in V_{X\mathfrak{P}}$ such that

$$B(u_{X\mathfrak{P}},v) = F(v) \quad \forall\, v\in V_{X\mathfrak{P}}. \qquad (2)$$

Hierarchical error estimation

The idea is to compare $u_{X\mathfrak{P}}\in V_{X\mathfrak{P}}$ with a second approximation $u^*_{X\mathfrak{P}}\in V^*_{X\mathfrak{P}} \supset V_{X\mathfrak{P}}$ supposed to be more accurate, so that the energy norm of the true error can be approximated by

$$\|e\|_B = \|u - u_{X\mathfrak{P}}\|_B \approx \|u^*_{X\mathfrak{P}} - u_{X\mathfrak{P}}\|_B =: \|e^*\|_B$$

where $e^*\in V^*_{X\mathfrak{P}}$ satisfies $B(e^*,v) = F(v) - B(u_{X\mathfrak{P}},v)$, $\forall\, v\in V^*_{X\mathfrak{P}}$. In order to avoid the expensive computation of $u^*_{X\mathfrak{P}}$, one defines $V^*_{X\mathfrak{P}}$ by means of the following **hierarchical enrichments**:

$$V^*_{X\mathfrak{P}} := V_{X\mathfrak{P}} \ \oplus\ \big(Y\ \otimes\ \mathcal{P}_{\mathfrak{P}}\big)\ \oplus\ \big(X\ \otimes\ \mathcal{P}_{\mathfrak{Q}}\big) = V_{X\mathfrak{P}}\ \oplus\ V_{Y\mathfrak{P}}\ \oplus\ V_{X\mathfrak{Q}}$$

where $Y\subset H_0^1(D)$ is a **detail space** such that $X\cap Y = \{0\}$ (with $X^* := X\oplus Y$), and $\mathfrak{Q}\subset\mathfrak{I}$ is a **detail index set** such that $\mathfrak{P}\cap\mathfrak{Q} = \emptyset$ (with $\mathfrak{P}^* := \mathfrak{P}\cup\mathfrak{Q}$).

By exploiting the hierarchical decomposition of $V^*_{X\mathfrak{P}}$, we let the "spatial" estimator $e_{Y\mathfrak{P}}\in V_{Y\mathfrak{P}}$ and the "parametric" estimator $e_{X\mathfrak{Q}}\in V_{X\mathfrak{Q}}$ satisfying respectively

$$B_0(e_{Y\mathfrak{P}},v) = F(v) - B(u_{X\mathfrak{P}},v) \quad \forall\, v\in V_{Y\mathfrak{P}},$$

$$B_0(e_{X\mathfrak{Q}},v) = F(v) - B(u_{X\mathfrak{P}},v) \quad \forall\, v\in V_{X\mathfrak{Q}},$$

(we use the the fact that $\|\cdot\|_B \simeq \|\cdot\|_{B_0}$) and we define the following global error estimate:

$$\eta := \big(\ \|e_{Y\mathfrak{P}}\|^2_{B_0} + \|e_{X\mathfrak{Q}}\|^2_{B_0}\ \big)^{1/2}.$$

(i) The error estimate is **reliable and efficient** [2], i.e., $\eta\ \simeq\ \|u - u_{X\mathfrak{P}}\|_B = \|e\|_B$.
(ii) The quantities $\|e_{X\mathfrak{Q}}\|_{B_0}$ and $\|e_{Y\mathfrak{P}}\|_{B_0}$ provide an estimate of the **error reduction** achieved if either the enanched approximation $u_{X^*\mathfrak{P}}\in V_{X^*\mathfrak{P}}$ or $u_{X\mathfrak{P}^*}\in V_{X\mathfrak{P}^*}$ is going to be computed, i.e, $\|e_{Y\mathfrak{P}}\|_{B_0}\ \simeq\ \|u_{X^*\mathfrak{P}} - u_{X\mathfrak{P}}\|_B$ and $\|e_{X\mathfrak{Q}}\|_{B_0}\ \simeq\ \|u_{X\mathfrak{P}^*} - u_{X\mathfrak{P}}\|_B$ [2].

Error-reduction based adaptive algorithm

> Input: data a, f; coarse $\mathcal{T}_0$; initial index set $\mathfrak{P}_0$; parameters $\theta_X, \theta_{\mathfrak{P}}\in(0,1)$; tolerance ε;

1. SOLVE: compute $u_k = u^k_{X\mathfrak{P}}\in V^k_{X\mathfrak{P}}$ by solving the discrete formulation (2);
2. ESTIMATE: compute the estimate $\eta_k := \big(\ \sum_{K\in\mathcal{T}_k}\|e_{Y\mathfrak{P}}|_K\|^2_{B_0} + \sum_{\mu\in\mathfrak{Q}_k}\|e^{(\mu)}_{X\mathfrak{Q}}\|^2_{B_0}\ \big)^{1/2}$, where $e_{Y\mathfrak{P}}|_K$ and $e^{(\mu)}_{X\mathfrak{Q}}$ represent the error contributions from each element $K\in\mathcal{T}$ and each index $\mu\in\mathfrak{Q}$, respectively;

If $\eta_k < \varepsilon$ then **end loop**; otherwise:

3. MARK: using Dörfler marking, create a set of marked elements $\mathcal{M}_k\subset\mathcal{T}_k$ and a set of marked indices $\mathfrak{M}_k\subset\mathfrak{Q}_k$ such that:

$$\eta^2_{\mathcal{M}} := \textstyle\sum_{K\in\mathcal{M}_k}\|e_{Y\mathfrak{P}}|_K\|^2_{B_0} \geq \theta_X\|e_{Y\mathfrak{P}}\|^2_{B_0} \qquad \text{and} \qquad \eta^2_{\mathfrak{M}} := \textstyle\sum_{\mu\in\mathfrak{M}_k}\|e^{(\mu)}_{X\mathfrak{Q}}\|^2_{B_0} \geq \theta_{\mathfrak{P}}\|e_{X\mathfrak{Q}}\|^2_{B_0}$$

4. REFINE: **if** $\eta_{\mathfrak{M}} > \eta_{\mathcal{M}}$ **then**
   then polynomial enrichment: $\mathfrak{P}_{k+1} = \mathfrak{P}_k\cup\mathfrak{M}_k$ and set $\mathcal{T}_{k+1} = \mathcal{T}_k$,
   else mesh refinement: $\mathcal{T}_{k+1} = \text{longest-edge-bisection}(\mathcal{T}_k)$ and set $\mathfrak{P}_{k+1} = \mathfrak{P}_k$;

> Output: final Galerkin solution u_k; final error estimate η_k;

Numerical experiment

We consider the single parameters y_m to be independent and identically distributed uniformly in $(-1,1)$, i.e., $y_m\sim U(-1,1)$. This corresponds to single measures $\pi_m(y_m) = 1/2$.

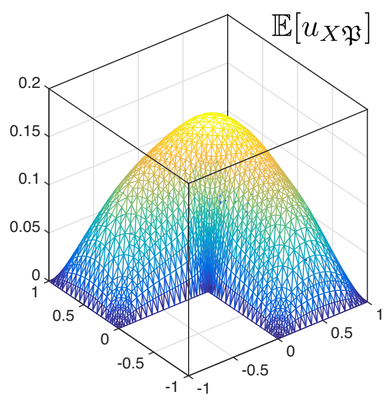
Let the problem (1) be posed on the L-shaped domain $D := (-1,1)^2\setminus(-1,0]^2$, with $f(\mathbf{x}) = 1$ and consider the following random coefficient [3]:

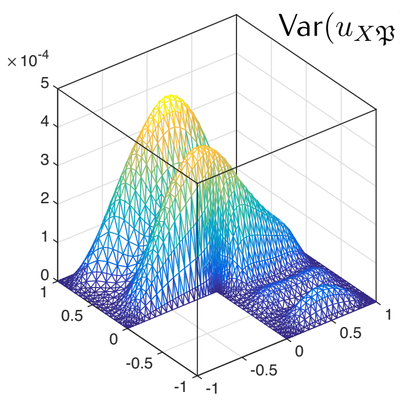
$$a(\mathbf{x},\mathbf{y}) = 1 + \textstyle\sum_{m\geq 1} y_m\left[\frac{\bar\alpha}{m^2}\cos(2\pi\beta_1(m)x_1)\cos(2\pi\beta_2(m)x_2)\right] \qquad \bar\alpha = 0.547,$$

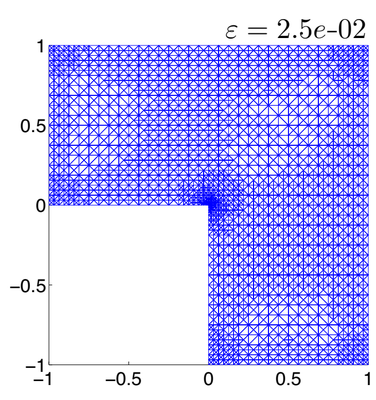
with

$$\beta_1(m) := m - k(m)(k(m)+1)/2, \quad \beta_2(m) := k(m) - \beta_1(m), \quad k(m) := \lfloor -1/2 + \sqrt{1/4+2m}\rfloor.$$

The following, is the result of running the adaptive algorithm using $\theta_X = 0.2, \theta_{\mathfrak{P}} = 0.8$ and $\varepsilon = 5e\text{-}03$. Below, it is plotted the mean value and the variance of the computed stochastic Galerkin solution and the spatial mesh when an intermediate tolerance was met. Due to the smaller magnitude of the variance, we observe how the spatial refinement is mostly driven by the (singular) behaviour of the mean in the corner at the origin. Overall, the algorithm took 419 sec. and $K = 63$ iterations to reach the tolerance, with a final number of dofs equal to $N_K = \dim(X_K)\cdot\#\mathfrak{P}_K = 603\text{'}594$, and $\#\mathfrak{P}_K = 18$ total indices.





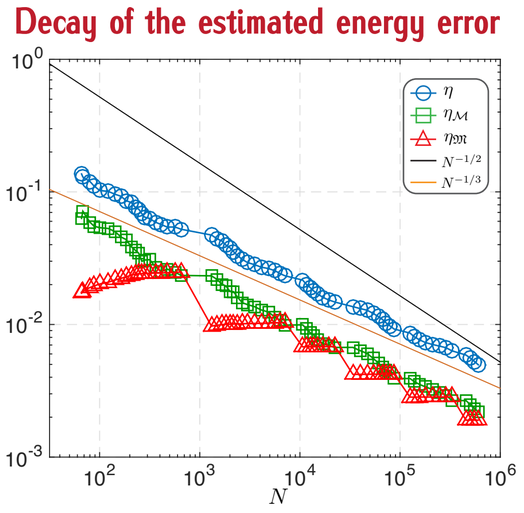


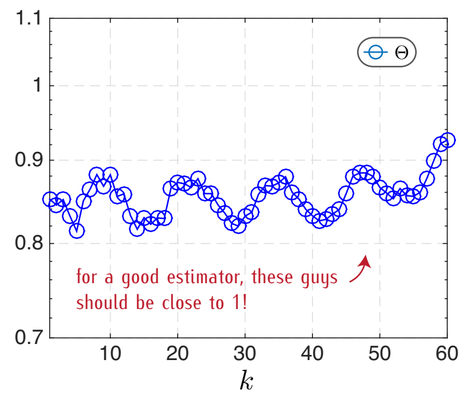
Final index set $\#\mathfrak{P}_{63}$ (k denotes the iteration): active parameters 6, total pol. degree 4;

$k=0$	(0 0)	$k=52$	(0 0 0 0 1)
	(1 0)		(1 0 0 1 0)
$k=20$	(0 1)		(2 0 1 0 0)
	(2 0)		(2 1 0 0 0)
$k=35$	(0 0 1)	$k=60$	(0 0 0 0 0 1)
	(1 1 0)		(0 2 0 0 0 0)
$k=43$	(0 0 0 1)		(1 0 0 0 1 0)
	(1 0 1 0)		(3 1 0 0 0 0)
	(3 0 0 0)		(4 0 0 0 0 0)

Using a very accurate solution u_{ref}, we test the quality of the estimator by computing the **effectivity indices** $\Theta_k := \eta_k/(\|u_{\text{ref}}\|^2_B - \|u_{X\mathfrak{P}}\|^2_B)^{1/2}$ for each k. Overall, the true error is well (under)estimated.

Decay of the estimated energy error





Conclusion

In our adaptive algorithm, the error is controlled via hierarchichal error estimates and we use separate precise error reduction estimates for both spatial and stochastic approximations to guide the adaptive refinement. A balance between the two sources of error is then guaranteed and this make the algorithm computationally efficient and (numerically!) convergent.

References

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[2] **A. Bespalov and D. Silvester**, SIAM J. Sci. Comput., 38 (2016), pp. A2118-A2140;
[3] **M. Eigel, C. J. Gittelson, C. Schwab, and E. Zander**, ESAIM Math. Model. Numer. Anal., 49 (2015), pp. 1367-1398;