Efficient adaptive algorithms for PDEs with random input data

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joint work with Alex Bespalov

LAMS seminar

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Outline

1. Stochastic Galerkin FEM
2. Hierarchical a posteriori error estimation
3. Adaptive algorithm
4. Some numerical examples

1. Stochastic Galerkin FEM

Parametric model problem

We are interested in the following steady-state parametric diffusion problem:

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\begin{aligned}
-\nabla \cdot (a(\mathbf{x}, \mathbf{y}) \nabla u(\mathbf{x}, \mathbf{y})) &= f(\mathbf{x}) \quad && \mathbf{x} \in D, \quad \mathbf{y} \in \Gamma, \\
u(\mathbf{x}, \mathbf{y}) &= 0 && \mathbf{x} \in \partial D, \ \mathbf{y} \in \Gamma,
\end{aligned}
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- $D \subset \mathbb{R}^2$ is a spatial bounded domain,
- the *parameter domain* Γ is given by $\Gamma := \prod_{m=1}^{\infty} \Gamma_m$, with $\mathbf{y} = (y_1, y_2, \dots) \in \Gamma$ being a multivariate-random parameter, $y_m \in \Gamma_m \subset \mathbb{R}$,

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and $a(\mathbf{x}, \mathbf{y})$ is a parametric diffusion coefficient which we suppose to be given by

$$
a(\mathbf{x}, \mathbf{y}) = a_0(\mathbf{x}) + \sum_{m=1}^{\infty} y_m \, a_m(\mathbf{x}), \qquad \mathbf{x} \in D, \ \mathbf{y} \in \Gamma,
$$

where the smooth functions $\{a_m\} \in L^\infty(D)$, for all m, satisfy the following conditions:

- positivity: $0 < \alpha_0^{\min} \leq a_0(\mathbf{x}) \leq \alpha_0^{\max}$, a.e. in D,
- convergence: $\sum_{m=1}^{\infty} \|a_m\|_{L^\infty(D)} < \alpha_0^{\min}$.

"Modelling" the uncertainty

In order to deal with the parameter $\mathbf{y} \in \Gamma$, we will need a measure defined on the parameter domain Γ. We will assume the following:

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- for each m, we let $\pi_m : \Gamma_m \to \mathbb{R}^+$ be a *distribution measure* on $(\Gamma_m, \mathcal{B}(\Gamma_m))$ s.t.

$$\pi(\mathbf{y}) := \prod_{m=1}^{\infty} \pi_m(y_m), \quad \mathbf{y} \in \Gamma,$$

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  is a *product distribution measure* on $(\Gamma, \mathcal{B}(\Gamma))$.
- if we assume π absolutely continuous with respect to the Lebesgue measure, then $d\pi(\mathbf{y}) = \rho(\mathbf{y})d\mathbf{y}$.

Continuous stochastic weak formulation

Define the tensor product space V as

$$V := H_0^1(D) \otimes L_\pi^2(\Gamma),$$

where

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Then, we look for a function $u = u(\mathbf{x}, \mathbf{y}) \in V$ such that

$$B(u, v) := B_0(u, v) + \sum_{m=1}^{\infty} B_m(u, v) = \int_\Gamma \int_D f(\mathbf{x}) v(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\pi(\mathbf{y}) =: F(v),$$

for all $v \in V$, where

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- Assumptions on $a(\mathbf{x}, \mathbf{y})$ ensure the well-posedness of the weak formulation;
- The bilinear forms B and B_0 induce two equivalent norms

$$B(\cdot, \cdot) =: \|\cdot\|_B^2 \simeq \|\cdot\|_{B_0}^2 := B_0(\cdot, \cdot).$$

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The discretisation of the weak formulation requires the definition of a finite dimensional subspace of $V = H_0^1(D) \otimes L_\pi^2(\Gamma)$.

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$$V_{X\mathcal{P}} := X \otimes \mathcal{P}_\mathcal{P} \quad \subset \quad V,$$

and we look for a function $u_{X\mathcal{P}} \in V_{X\mathcal{P}}$ such that

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Spatial discretisation is done using FEM:

- we let $X = X(\mathcal{T}) \subset H_0^1(D)$ be a first-order Finite Element space, that is, it is spanned by piecewise linear functions associated with a mesh $\mathcal{T}$ of D.

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What about the space $\mathcal{P}_\mathcal{P}$?

Discretisation of the stochastic component

We will let $\mathcal{P}_{\mathcal{P}} \subset L^2_\pi(\Gamma)$ be a polynomial space on Γ constructed as follows. Consider:

- an orthonormal basis $\{p_k\}_{k\geq 0}$ (with $p_0 := 1$) of $L^2_{\pi_m}(\Gamma_m)$, for each m;
- the set of *finitely-supported* sequences (indices)

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$$\mathcal{P} := \left\{ \nu_{(1)} = (0, 0, \dots), \ \nu_{(2)} = (1, 0, \dots), \ \nu_{(3)} = (0, 1, \dots), \ \nu_{(4)} = (2, 1, \dots) \right\}$$

$$\#\mathrm{supp}(\nu_{(1)}) = 0, \quad \#\mathrm{supp}(\nu_{(2)}) = 1, \quad \#\mathrm{supp}(\nu_{(3)}) = 1, \quad \#\mathrm{supp}(\nu_{(4)}) = 2$$

$$P_{\nu_{(1)}} = 1, \quad P_{\nu_{(2)}} = p_1(y_1) \quad P_{\nu_{(3)}} = p_1(y_2) \quad P_{\nu_{(4)}} = p_2(y_1)\, p_1(y_2)$$

$$\Longrightarrow \quad \mathcal{P}_{\mathcal{P}} = \mathrm{span}\{ \ 1, \ p_1(y_1), \ p_1(y_2), \ p_2(y_1)\, p_1(y_2) \ \}$$

Discretisation of the stochastic component

We will let $\mathcal{P}_\mathcal{P} \subset L^2_\pi(\Gamma)$ be a polynomial space on Γ constructed as follows. Consider:

- an orthonormal basis $\{p_k\}_{k\geq 0}$ (with $p_0 := 1$) of $L^2_{\pi_m}(\Gamma_m)$, for each m;
- the set of *finitely-supported* sequences (indices)

$$\mathcal{J} := \left\{ \nu = (\nu_1, \nu_2, \dots) \in \mathbb{N}_0^{\mathbb{N}} \ : \ \#\mathrm{supp}(\nu) < \infty \right\}.$$

Now, pick a finite-dimensional set of indices $\mathcal{P} \subset \mathcal{J}$ and define:

$$\mathcal{P}_\mathcal{P} := \mathrm{span}\left\{ P_\nu(\mathbf{y}) := \prod_{m=1}^{\infty} p_{\nu_m}(y_m) \ : \ \nu \in \mathcal{P} \right\}.$$

Example

$$\mathcal{P} := \left\{ \nu_{(1)} = (0,0,\dots),\ \nu_{(2)} = (1,0,\dots),\ \nu_{(3)} = (0,1,\dots),\ \nu_{(4)} = (2,1,\dots) \right\}$$

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$$P_{\nu_{(1)}} = 1, \quad P_{\nu_{(2)}} = p_1(y_1) \quad P_{\nu_{(3)}} = p_1(y_2) \quad P_{\nu_{(4)}} = p_2(y_1)\, p_1(y_2)$$

$$\Longrightarrow \quad \mathcal{P}_\mathcal{P} = \mathrm{span}\{\ 1,\ p_1(y_1),\ p_1(y_2),\ p_2(y_1)\, p_1(y_2)\ \}$$

2. Hierarchical a posteriori error estimation

Hierarchical enrichments

Task: estimate the accuracy of $u_{X\mathcal{P}} \in V_{X\mathcal{P}}$.

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Idea: compare $u_{X\mathcal{P}} \in V_{X\mathcal{P}}$ with another approximation $u^*_{X\mathcal{P}} \in V^*_{X\mathcal{P}} \supset V_{X\mathcal{P}}$ that is supposed to be more accurate. Therefore

$$\|e\|_B = \|u - u_{X\mathcal{P}}\|_B \approx \|u^*_{X\mathcal{P}} - u_{X\mathcal{P}}\|_B =: \|e^*\|_B \quad \leftarrow \quad \text{error estimate}$$

where e^* satisfies

$$B(e^*, v) = F(v) - B(u_{X\mathcal{P}}, v) \qquad \forall\, v \in V^*_{X\mathcal{P}}.$$

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Smarter idea: to avoid the computation of $u^*_{X\mathcal{P}}$, consider the following steps:

- take a "detail space" $Y \subset H^1_0(D)$ such that $X \cap Y = \{0\}$, and define:

$$X^* := X \oplus Y;$$

- take a "detail index set" $\mathcal{Q} \subset \mathcal{I}$ such that $\mathcal{P} \cap \mathcal{Q} = \emptyset$, and define:

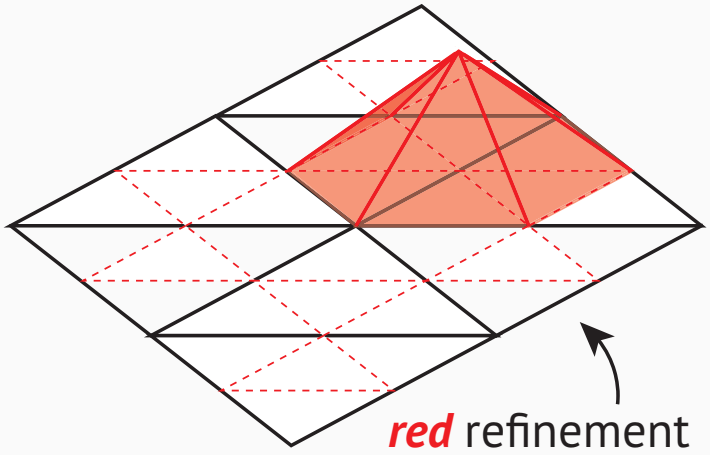
$$\mathcal{P}_{\mathcal{P}^*} := \mathcal{P}_{\mathcal{P}} \oplus \mathcal{P}_{\mathcal{Q}}, \qquad \mathcal{P}^* := \mathcal{P} \cup \mathcal{Q};$$

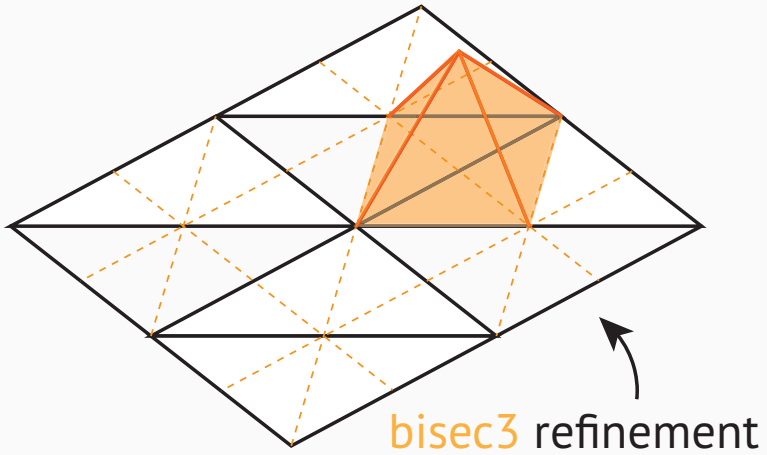
- finally, define the enhanced space as

$$V^*_{X\mathcal{P}} := V_{X\mathcal{P}} \oplus \underbrace{\left(Y \otimes \mathcal{P}_{\mathcal{P}}\right)}_{=:V_{Y\mathcal{P}}} \oplus \underbrace{\left(X \otimes \mathcal{P}_{\mathcal{Q}}\right)}_{=:V_{X\mathcal{Q}}}.$$

Examples of detail space and detail index set

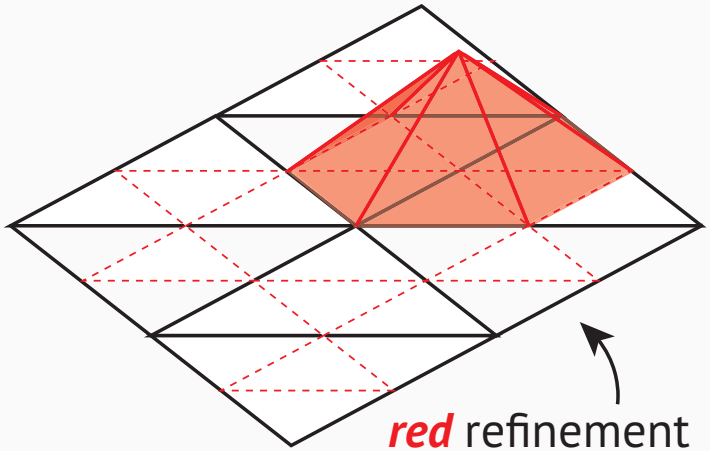
- Typically, the detail space Y is a first-order Finite Element space *spanned* by the linear functions associated with the edge midpoints of $\mathcal{T}$, that is,

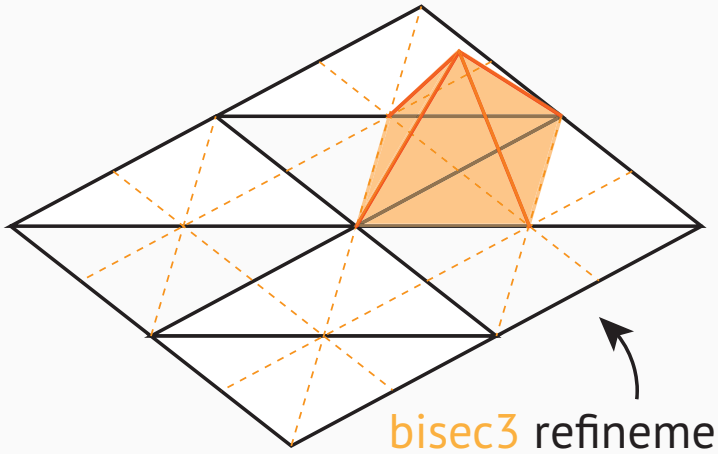




Examples of detail space and detail index set

- Typically, the detail space $\mathcal{Y}$ is a first-order Finite Element space *spanned* by the linear functions associated with the edge midpoints of $\mathcal{T}$, that is,





- We construct the detail index set $\mathcal{Q}$ from $\mathcal{P}$ by activating 1 parameter more as well as increasing the total polynomial degree in the active parameters by 1. For example:

$$\text{if} \quad \mathcal{P} = \left\{ \begin{array}{l} (0,0,0,\dots) \\ (1,0,0,\dots) \end{array} \right\} \quad \text{then} \quad \mathcal{Q} = \left\{ \begin{array}{l} (0,1,0,\dots) \\ (1,1,0,\dots) \\ (2,0,0,\dots) \end{array} \right\}.$$

Hierarchical error estimates

By exploiting the hierarchical decomposition of $V^*_{X\mathcal{P}}$, we define the "spatial" estimator $e_{Y\mathcal{P}} \in V_{Y\mathcal{P}}$ and the "parametric" estimator $e_{X\mathcal{Q}} \in V_{X\mathcal{Q}}$ satisfying respectively

$$B_0(e_{Y\mathcal{P}}, v) = F(v) - B(u_{X\mathcal{P}}, v) \qquad \forall\, v \in V_{Y\mathcal{P}},$$
$$B_0(e_{X\mathcal{Q}}, v) = F(v) - B(u_{X\mathcal{P}}, v) \qquad \forall\, v \in V_{X\mathcal{Q}},$$

and let

$$\eta := \left(\|e_{Y\mathcal{P}}\|^2_{B_0} + \|e_{X\mathcal{Q}}\|^2_{B_0} \right)^{1/2}.$$

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- The error estimate η is both *reliable* and *efficient*, that is,

$$\eta \simeq \|u - u_{X\mathcal{P}}\|_B = \|e\|_B;$$

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- The error estimate η is both *reliable* and *efficient*, that is,

$$\eta \simeq \|u - u_{X\mathcal{P}}\|_B = \|e\|_B;$$

- The quantities $\|e_{Y\mathcal{P}}\|_{B_0}$ and $\|e_{X\mathcal{Q}}\|_{B_0}$ provide an estimate of the error reduction achieved,

$$\|e_{Y\mathcal{P}}\|_{B_0} \simeq \|u_{X^*\mathcal{P}} - u_{X\mathcal{P}}\|_B, \qquad \|e_{X\mathcal{Q}}\|_{B_0} \simeq \|u_{X\mathcal{P}^*} - u_{X\mathcal{P}}\|_B,$$

if either $u_{X^*\mathcal{P}} \in V_{X\mathcal{P}} \oplus V_{Y\mathcal{P}}$ or $u_{X\mathcal{P}^*} \in V_{X\mathcal{P}} \oplus V_{X\mathcal{Q}}$ is going to be computed.

3. Adaptive algorithm

Error-reduction based adaptive algorithm

A typical adaptive Finite Element method makes use of the following loop

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}.$$

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- **Input**: data a, f; coarse $\mathcal{T}_0$; initial $\mathcal{P}_0$; parameters $\theta_{\mathcal{X}}, \theta_{\mathcal{Q}} \in (0, 1)$; tolerance ε;
- **Output**: final discrete solution u_k; final energy error estimate η_k;

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1. SOLVE: compute u_k s.t. $B(u_k, v) = F(v)$, for all $v \in V^k_{\mathcal{X}\mathcal{P}} = \mathcal{X}(\mathcal{T}_k) \otimes \mathcal{P}_{\mathcal{P}_k}$;

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2. ESTIMATE: let $e^{(K)}_{\mathcal{Y}\mathcal{P}}$ and $e^{(\mu)}_{\mathcal{X}\mathcal{Q}}$ be the *local* error contributions from each element $K \in \mathcal{T}_k$ and each single index $\mu \in \mathcal{Q}_k$, respectively. Then, compute the error estimate

$$\eta_k = \left(\eta^2_{\mathcal{T}} + \eta^2_{\mathcal{Q}} \right)^{1/2} = \left(\sum_{K \in \mathcal{T}_k} \| e^{(K)}_{\mathcal{Y}\mathcal{P}} \|^2_{B_0} + \sum_{\mu \in \mathcal{Q}_k} \| e^{(\mu)}_{\mathcal{X}\mathcal{Q}} \|^2_{B_0} \right)^{1/2}.$$

If $\eta_k < \varepsilon$ then end loop; otherwise:

Error-reduction based adaptive algorithm

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- **Input**: data a, f; coarse $\mathcal{T}_0$; initial $\mathcal{P}_0$; parameters $\theta_{\mathcal{X}}, \theta_{\mathcal{Q}} \in (0,1)$; tolerance ε;
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$$\eta_k = \left(\eta^2_{\mathcal{T}} + \eta^2_{\mathcal{Q}} \right)^{1/2} = \left(\sum_{K \in \mathcal{T}_k} \| e^{(K)}_{\mathcal{Y}\mathcal{P}} \|^2_{B_0} + \sum_{\mu \in \mathcal{Q}_k} \| e^{(\mu)}_{\mathcal{X}\mathcal{Q}} \|^2_{B_0} \right)^{1/2}.$$

If $\eta_k < \varepsilon$ then end loop; otherwise:

3. MARK: create minimal sets of marked elements $\mathcal{M}_k \subseteq \mathcal{T}_k$ and marked indices $\mathcal{M}_k \subseteq \mathcal{Q}_k$ such that

$$\eta^2_{\mathcal{M}} := \sum_{K \in \mathcal{M}_k} \| e^{(K)}_{\mathcal{Y}\mathcal{P}} \|^2_{B_0} \geq \theta_{\mathcal{X}} \| e_{\mathcal{Y}\mathcal{P}} \|^2_{B_0}, \qquad \eta^2_{\mathcal{M}} := \sum_{\mu \in \mathcal{M}_k} \| e^{(\mu)}_{\mathcal{X}\mathcal{Q}} \|^2_{B_0} \geq \theta_{\mathcal{Q}} \| e_{\mathcal{X}\mathcal{Q}} \|^2_{B_0}.$$

Error-reduction based adaptive algorithm (cont.)

4. REFINE: we have to decide whether to perform a *spatial* refinement or a *parametric* polynomial enrichment according to the better error reduction provided.

Two cases:

- **Version 1** compares the *global* estimates $\eta_{\mathcal{T}}$ and $\eta_{\mathcal{Q}}$.
- **Version 2** compares the estimates $\eta_{\mathcal{M}}$ and $\eta_{\mathcal{M}}$;

Error-reduction based adaptive algorithm (cont.)

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Then we will do:

if $\eta_{\mathcal{T}} > \eta_{\mathcal{Q}}$ (Version 1) or $\eta_{\mathcal{M}} > \eta_{\mathcal{M}}$ (Version 2)

we enrich the spatial space X and keep the current $\mathcal{P}_{\mathcal{P}}$, that is,

$$\mathcal{T}_{k+1} = \text{refine}(\mathcal{T}_k, \mathcal{M}_k) \quad \text{and} \quad \mathcal{P}_{k+1} = \mathcal{P}_k;$$

otherwise

we keep the same X and enrich the parametric space $\mathcal{P}_{\mathcal{P}}$, that is,

$$\mathcal{T}_{k+1} = \mathcal{T}_k, \quad \text{and} \quad \mathcal{P}_{k+1} = \mathcal{P}_k \cup \mathcal{M}_k.$$

Error-reduction based adaptive algorithm (cont.)

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Initialisation:

- Parameters i.i.d. $y_m \sim U(-1,1)$: this corresponds to $\pi_m(y_m) = 1/2$, $m \geq 1$;
- The initial index set $\mathcal{P}_0$ has two indices and 1 active parameter:

$$\mathcal{P}_0 = \{(0,0,0,\dots), (1,0,0,\dots)\}.$$

4. Some numerical examples

Model Problem 1

Consider the problem on the square domain $D = (0,1)^2$,

$$\begin{aligned} -\nabla \cdot (a(\mathbf{x},\mathbf{y}) \nabla u(\mathbf{x},\mathbf{y})) &= 1 \qquad && \mathbf{x} = (x_1, x_2) \in D, \quad \mathbf{y} \in \Gamma, \\ u(\mathbf{x},\mathbf{y}) &= 0 && \mathbf{x} = (x_1, x_2) \in \partial D, \ \mathbf{y} \in \Gamma, \end{aligned}$$

where the random coefficient[1] is given by

$$a(\mathbf{x},\mathbf{y}) = 1 + \sum_{m=1}^{\infty} y_m \left[\frac{\bar{\alpha}}{m^2} \cos(2\pi\beta_1(m)x_1) \cos(2\pi\beta_2(m)x_2) \right], \qquad \bar{\alpha} = 0.547,$$

with

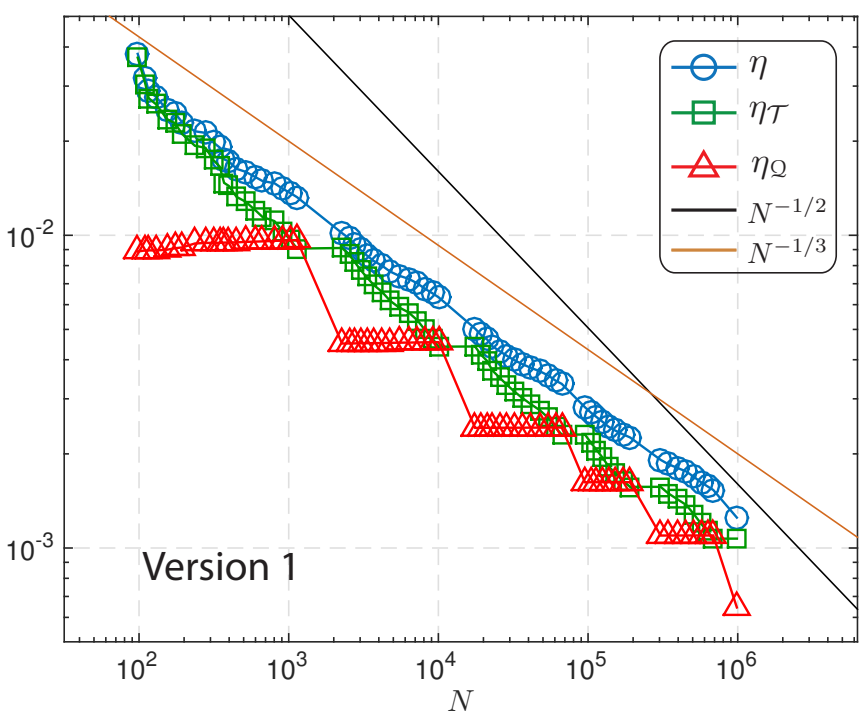
$$\begin{aligned} \beta_1(m) &:= m - k(m)(k(m)+1)/2, \\ \beta_2(m) &:= k(m) - \beta_1(m), \end{aligned} \qquad k(m) := \left\lfloor -1/2 + \sqrt{1/4 + 2m} \right\rfloor .$$

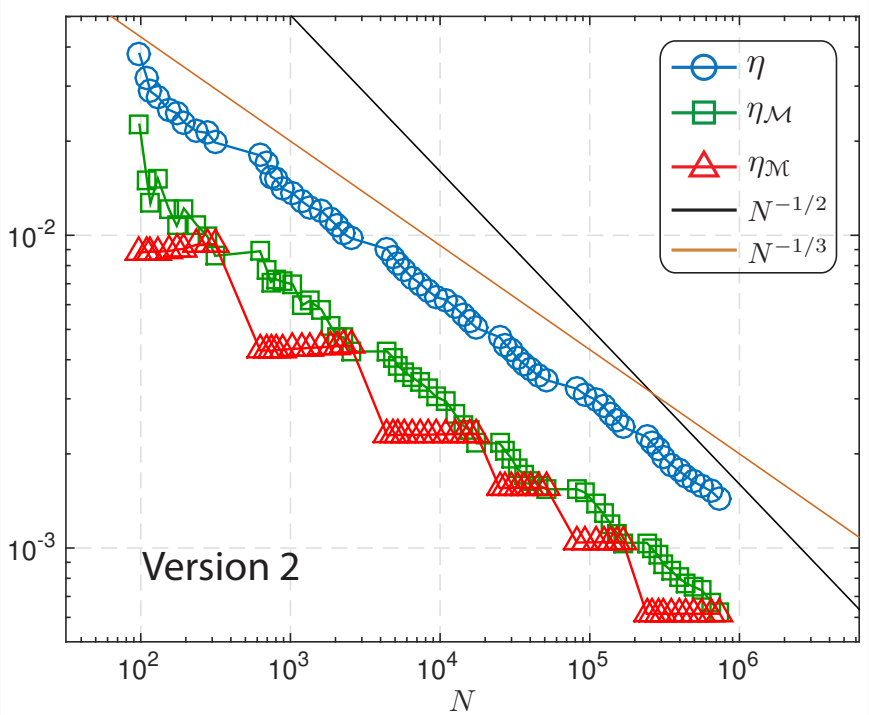
Algorithm's input parameters: $\theta_X = 0.2, \qquad \theta_{\mathcal{P}} = 0.9, \qquad \varepsilon = 1.5 \cdot 10^{-3}.$

[1] **Eigel, Gittelson, Schwab, Zander**, Comput. Methods. Appl. Mech. Engrg., 2014

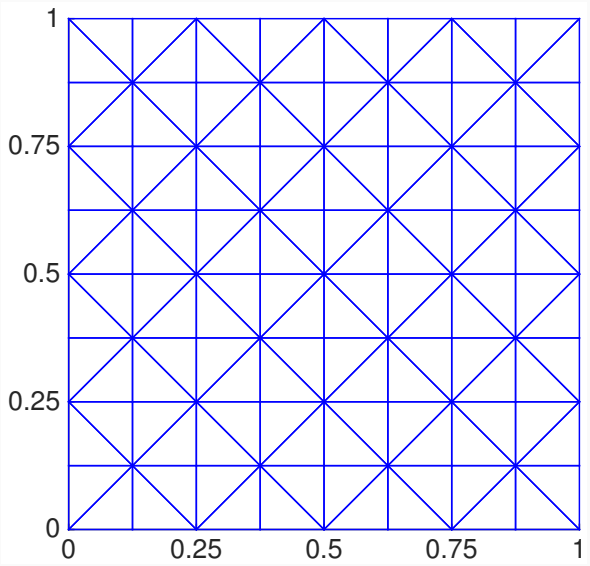
Model Problem 1: results

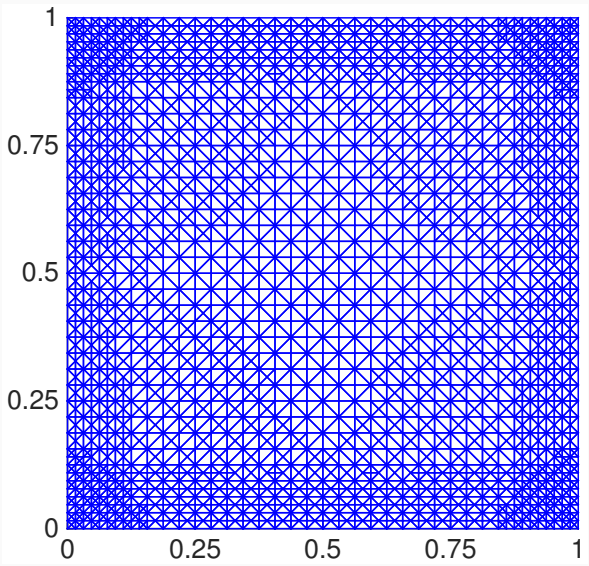
	Time (sec)	Iterations k	#Elements	#Indices	#Dofs
Version 1	605	64	86,552	23	986,769
Version 2	479	61	64,156	23	730,319

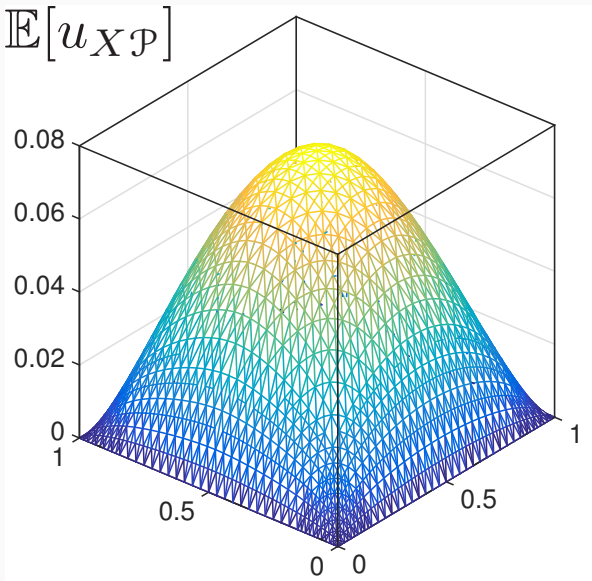


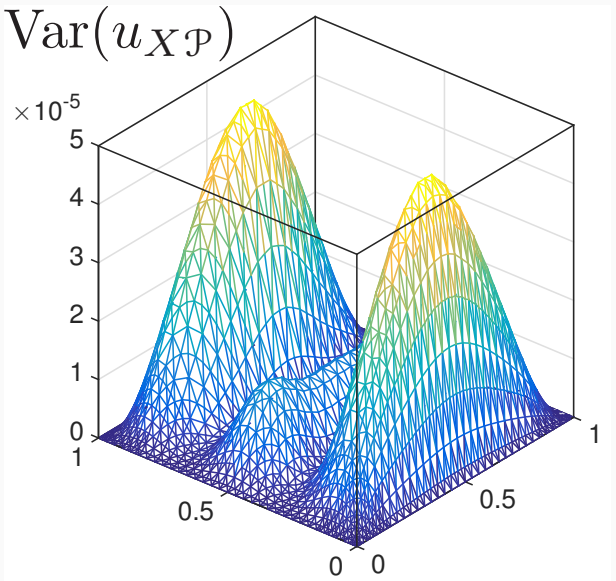


Model Problem 1: meshes, mean, and variance









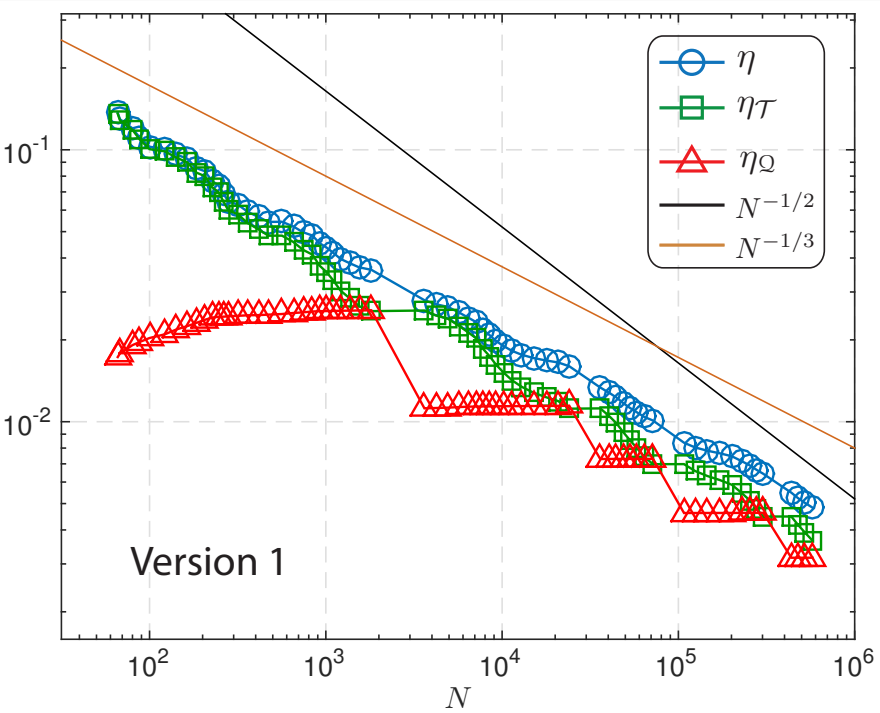
Model Problem 2

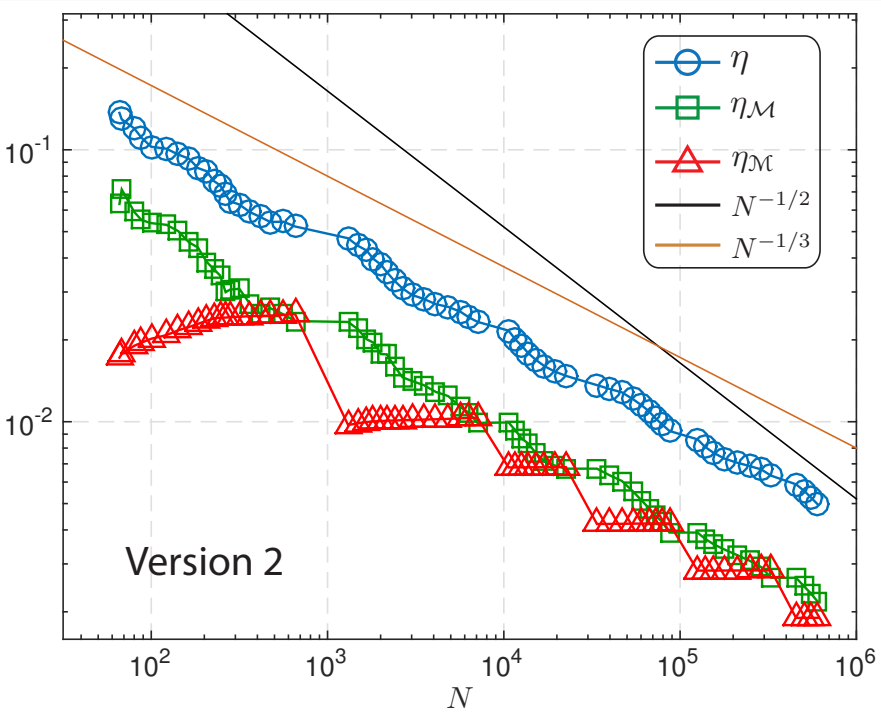
Consider again the same previous problem but this time posed on on the L-shaped domain $D := (-1,1)^2 \setminus (-1,0]^2$.

Algorithm's input parameters: $\theta_{\mathcal{X}} = 0.2, \qquad \theta_{\mathcal{P}} = 0.8, \qquad \varepsilon = 5 \cdot 10^{-3}.$

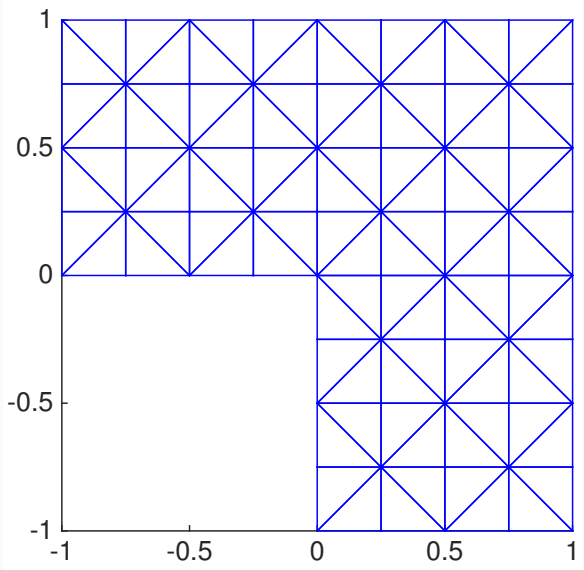
Model Problem 2: results

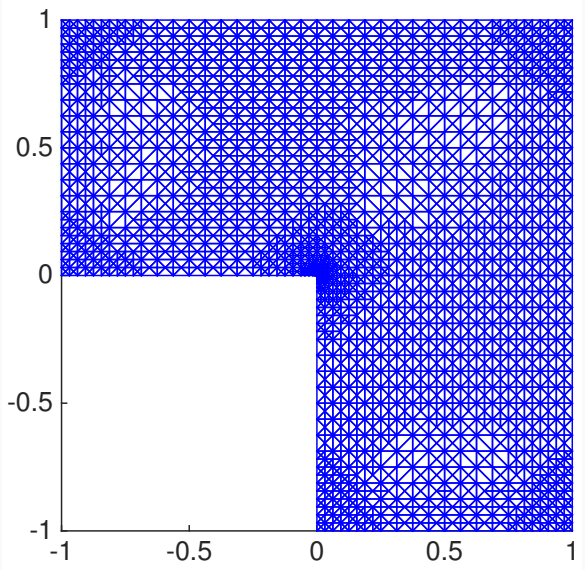
	Time (sec)	Iterations k	#Elements	#Indices	#Dofs
Version 1	476	65	89,480	13	576,121
Version 2	419	63	67,770	18	603,594

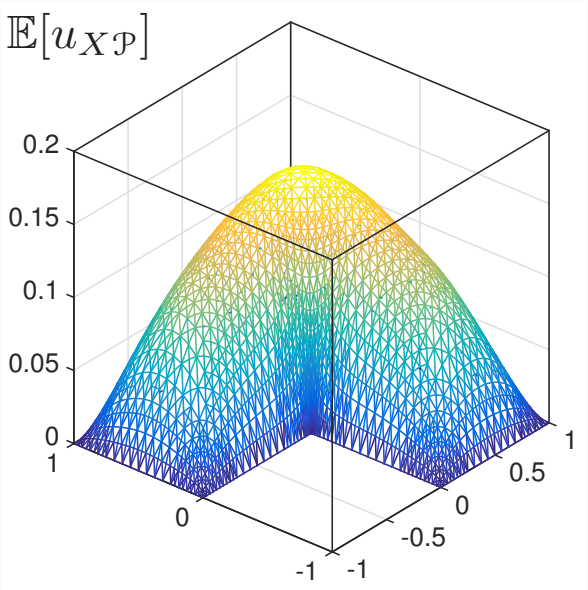


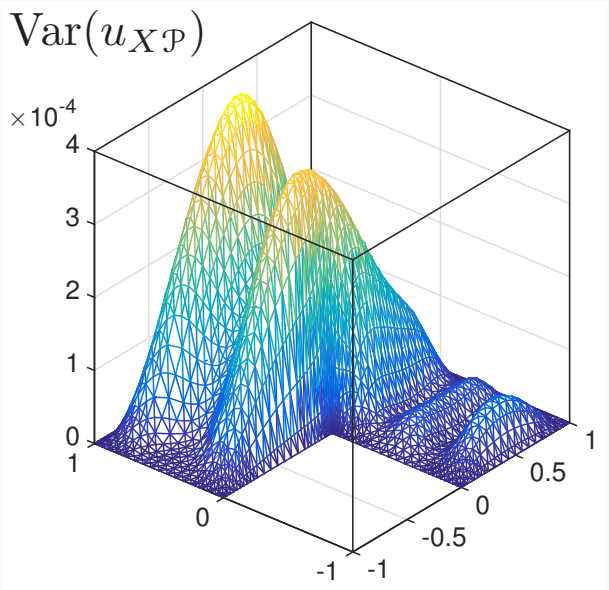


Model Problem 2: meshes, mean, and variance







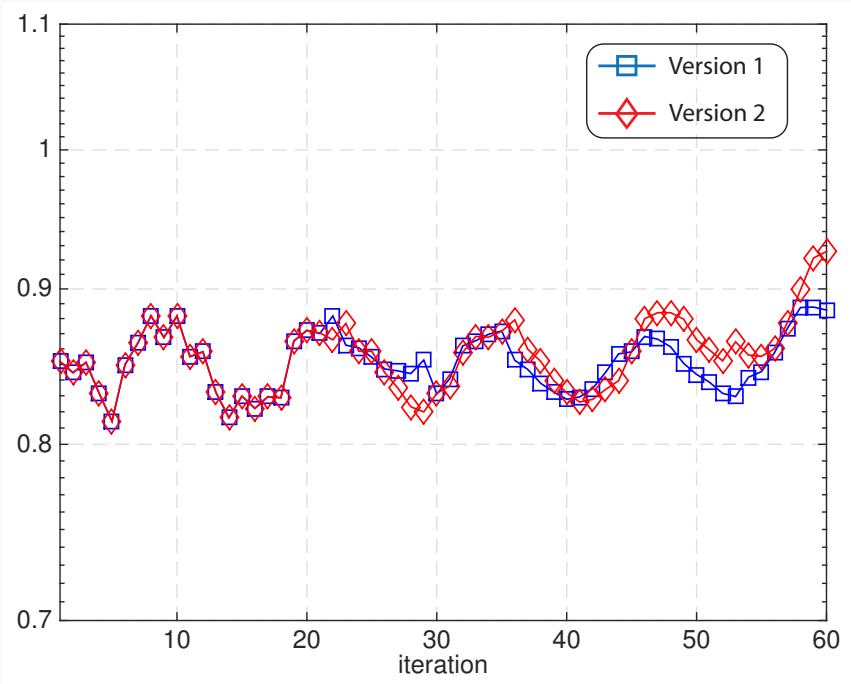


Model Problem 2: index sets and effectivity indices

	Version 1 $\mathcal{P}_{65}$	**Version 2** $\mathcal{P}_{63}$	
$k = 0$	(0 0) (1 0)	(0 0) (1 0)	$k = 0$
$k = 25$	(0 1) (2 0)	(0 1) (2 0)	$k = 20$
$k = 39$	(0 0 1) (1 1 0)	(0 0 1) (1 1 0)	$k = 35$
$k = 51$	(0 0 0 1) (1 0 1 0) (3 0 0 0)	(0 0 0 1) (1 0 1 0) (3 0 0 0)	$k = 43$
$k = 59$	(0 0 0 0 1) (1 0 0 1 0) (2 0 1 0 0) (2 1 0 0 0)	(0 0 0 0 1) (1 0 0 1 0) (2 0 1 0 0) (2 1 0 0 0)	$k = 52$
		(0 0 0 0 0 1) (0 2 0 0 0 0) (1 0 0 0 1 0) (3 1 0 0 0 0) (4 0 0 0 0 0)	$k = 60$

Effectivity indices computed with an accurate reference solution u_{ref} computed with P_2 Galerkin approximations ($\sim 27 \cdot 10^6 \, dofs$):

$$\Theta_k := \eta_k / \|u_{\text{ref}} - u_k\|_B$$



Model Problem 3

Consider the problem on the crack domain $D := (-1,1)^2 \setminus \{(-1,0) \times \{0\}\}$,

$$
\begin{aligned}
-\nabla \cdot \big(a(\mathbf{x},\mathbf{y}) \nabla u(\mathbf{x},\mathbf{y}) \big) &= e^{-(x_1+0.5)^2-(x_2-0.5)^2} && \mathbf{x} = (x_1,x_2) \in D, \quad \mathbf{y} \in \Gamma, \\
u(\mathbf{x},\mathbf{y}) &= 0 && \mathbf{x} = (x_1,x_2) \in \partial D, \ \mathbf{y} \in \Gamma.
\end{aligned}
$$

where the random coefficient is given by[1]

$$
a(\mathbf{x},\mathbf{y}) = 1 + \frac{1}{\sqrt{3}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} y_{ij} \sqrt{\nu_{ij}}\, \phi_{ij}(\mathbf{x}),
$$

with $\phi_{00}(\mathbf{x}) := 1$, $\nu_{00} := 1/4$, and

$$
\phi_{ij}(\mathbf{x}) := 2\cos(i\pi x_1)\cos(j\pi x_2), \qquad \nu_{ij} := (1/4)e^{-\pi(i^2+j^2)}, \qquad \forall\, i,j \geq 1.
$$

Reordering in terms of a single index m:

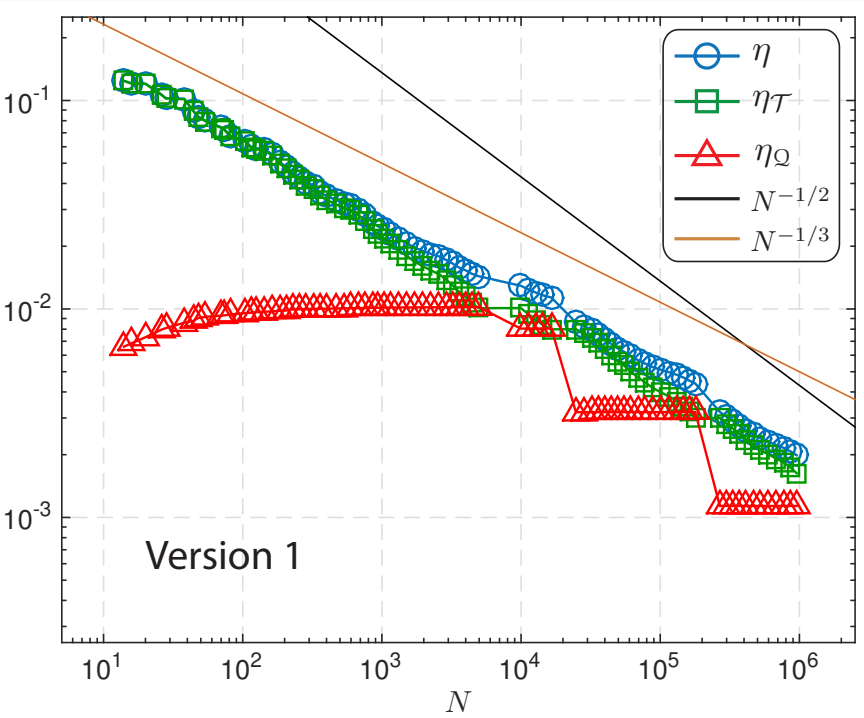
$$
a(\mathbf{x},\mathbf{y}) = 1 + \frac{1}{\sqrt{3}} \sum_{m=1}^{\infty} y_m \sqrt{\nu_m}\, \phi_m(\mathbf{x}).
$$

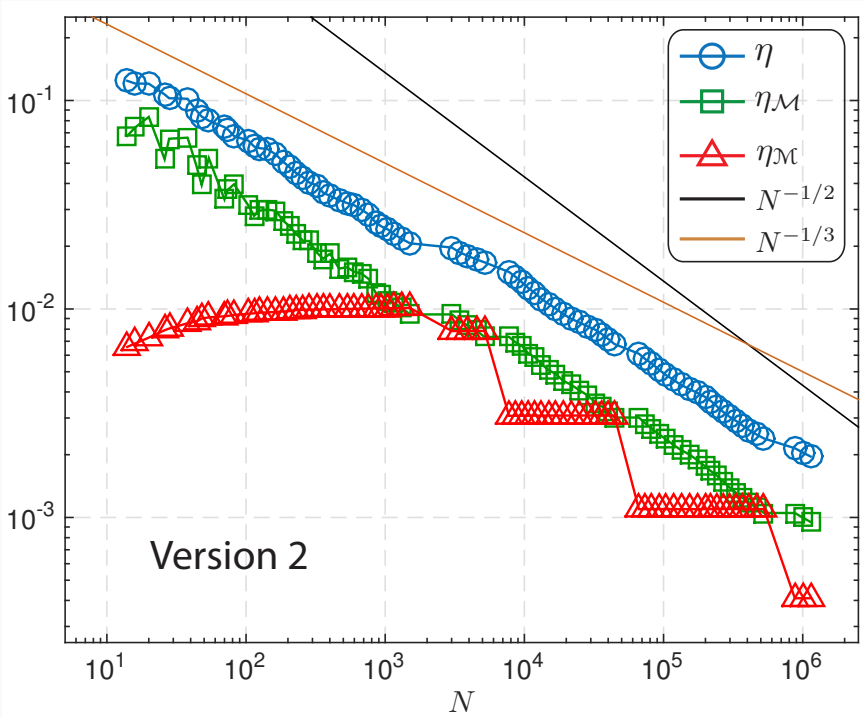
Algorithm's input parameters: $\theta_X = 0.2, \qquad \theta_{\mathcal{P}} = 0.9, \qquad \varepsilon = 2 \cdot 10^{-3}.$

[1] **Lord, Powell, Shardlow**, Cambridge University Press, 2014

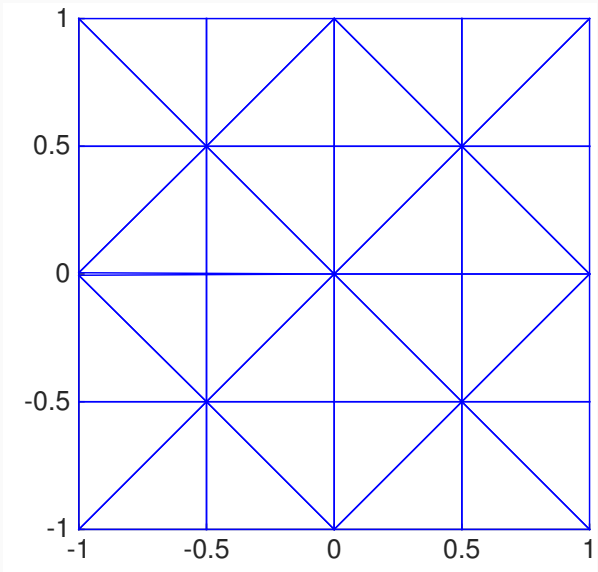
Model Problem 3: results

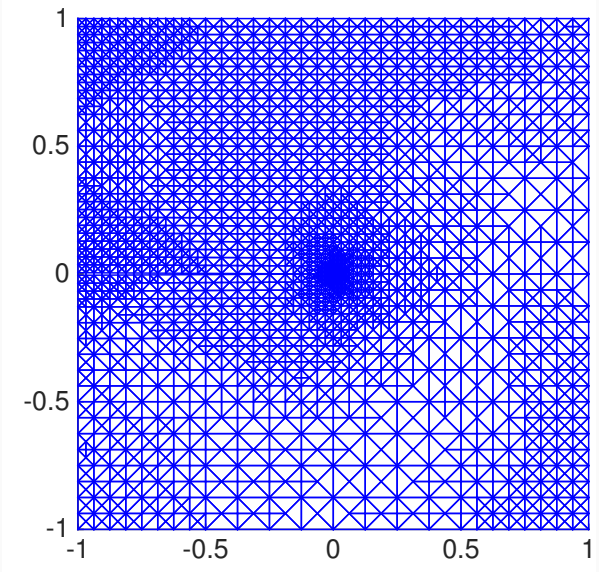
	Time (sec)	Iterations k	#Elements	#Indices	#Dofs
Version 1	1321	80	213,830	9	950,160
Version 2	935	78	153,312	15	1,140,720

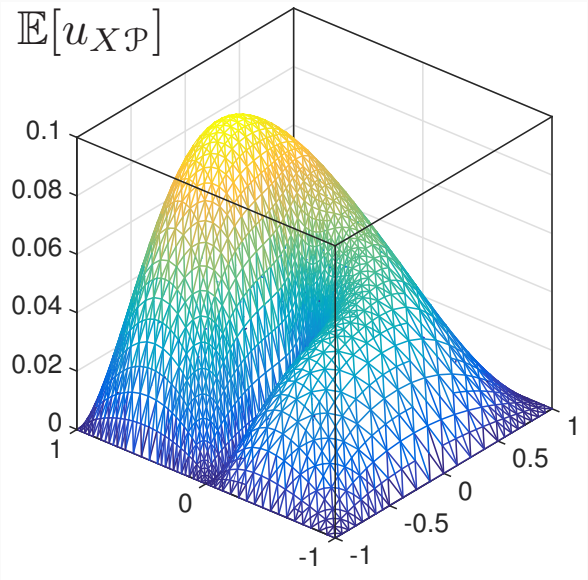


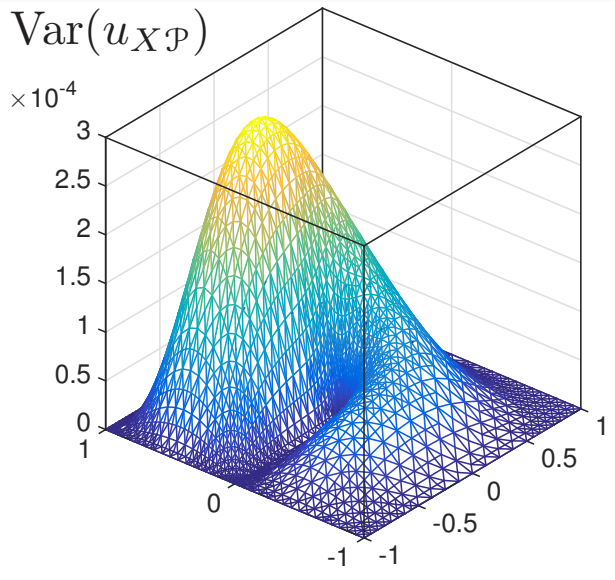


Model Problem 3: meshes, mean, and variance







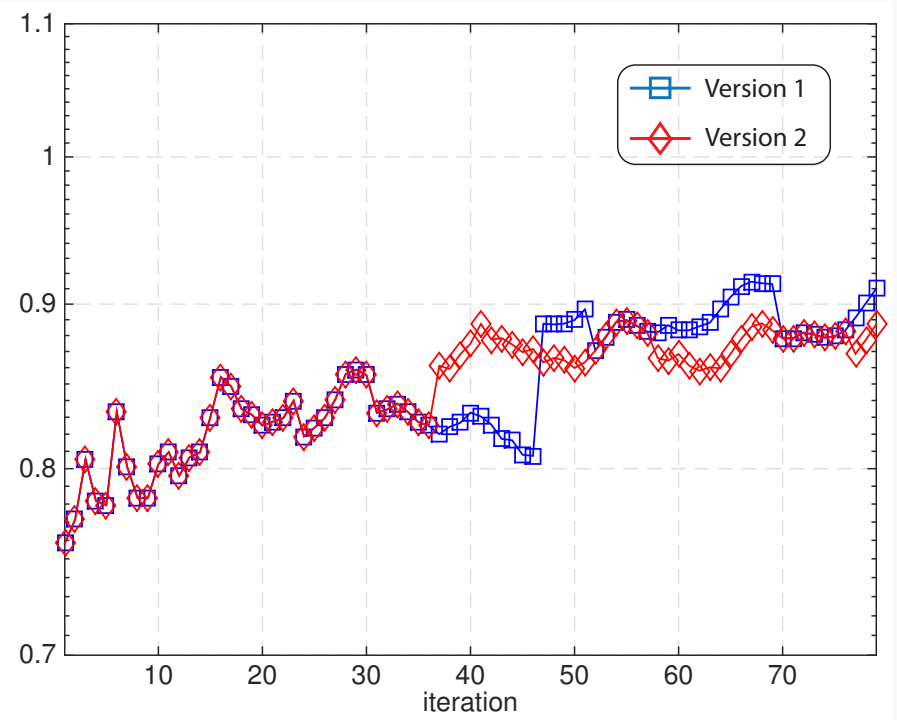


Model Problem 3: index sets and effectivity indices

	Version 1 $\mathcal{P}_{80}$	**Version 2** $\mathcal{P}_{78}$	
$k = 0$	(0 0) (1 0)	(0 0) (1 0)	$k = 0$
$k = 46$	(0 1) (2 0)	(0 1) (2 0)	$k = 36$
$k = 51$	(0 0 1) (1 1 0)	(0 0 1) (1 1 0)	$k = 41$
$k = 68$	(0 0 0 1) (1 0 1 0) (3 0 0 0)	(0 0 0 1) (1 0 1 0) (3 0 0 0)	$k = 57$
		(0 0 2 0) (0 1 1 0) (0 2 0 0) (1 0 0 1) (2 0 1 0) (2 1 0 0)	$k = 76$

Effectivity indices computed with an accurate reference solution u_{ref} computed with P_2 Galerkin approximations ($\sim 34{\cdot}10^6$ *dofs*):

$$\Theta_k := \eta_k / \|u_{\text{ref}} - u_k\|_B$$



Some remarks

- Two versions of the adaptive algorithm, based on the comparison of separate error reduction estimates in both spatial and parametric approximations;
- Efficient and convergent (numerically) algorithm for parametric problem with both regular and singular solutions;
- Optimal rates during mesh refinement steps provided appropriate threshold marking parameters.

Some remarks and future (current) work

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- Employing a new hierarchical *two-level* a posteriori error estimator in space $\mu_{\mathcal{Y}\mathcal{P}}$, which substitutes $e_{\mathcal{Y}\mathcal{P}}$ and speeds up the algorithm.

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- Employing a new hierarchical *two-level* a posteriori error estimator in space $\mu_{\mathcal{YP}}$, which substitutes $e_{\mathcal{YP}}$ and speeds up the algorithm.
- **Goal-oriented** adaptive algorithm for the approximation of quantity of interests $G(u)$ which are typically linear functionals of the solution u:

$$G(u) = \int_\Gamma \int_D g(\mathbf{x}) u(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\pi(\mathbf{y})$$

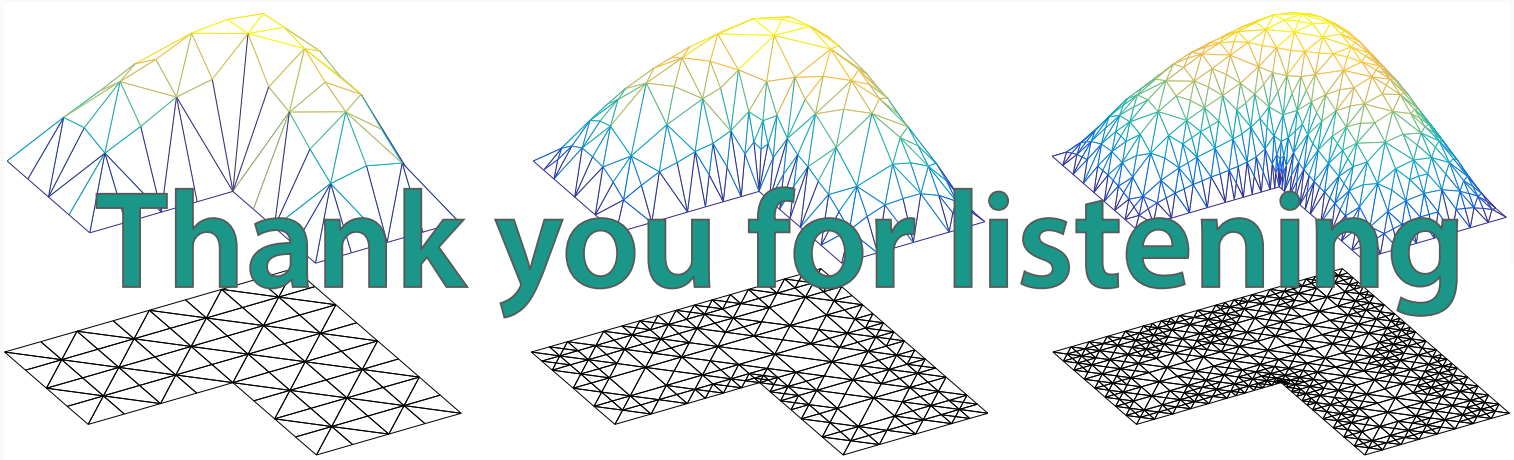
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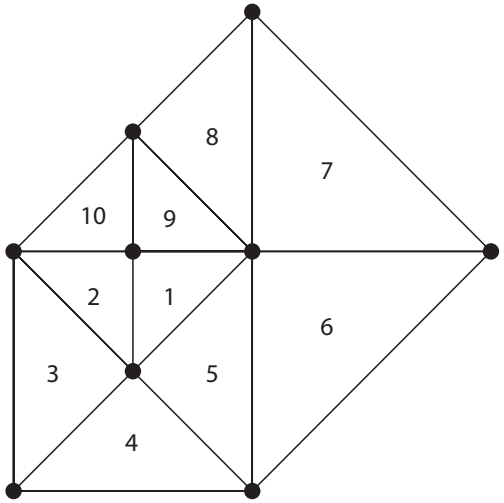
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- **C. Schwab and C. J. Gittelson**, *Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs*, Acta Numer., 20 (2011), pp. 291–467;
- **M. Eigel, C. J. Gittelson, C. Schwab, and E. Zander**, *A convergent adaptive stochastic Galerkin finite element method with quasi-optimal spatial meshes*, ESAIM Math. Model. Numer. Anal., 49 (2015), pp. 1367–1398;
- **A. Bespalov and D. Silvester**, *Efficient adaptive stochastic Galerkin methods for parametric operator equations*, SIAM J. Sci. Comput., 38 (2016), pp. A2118–A2140;
- **A. Bespalov and L. R.**, *Efficient adaptive algorithms for elliptic PDEs with random data*, SIAM/ASA J. Uncertain. Quantif., 6 (2018), pp. 243-272.
- **A. Bespalov and L. R.**, *Stochastic T-IFISS*, January 2018. Available on-line at http://web.mat.bham.ac.uk/A.Bespalov/software/index.html#stoch_tifiss.



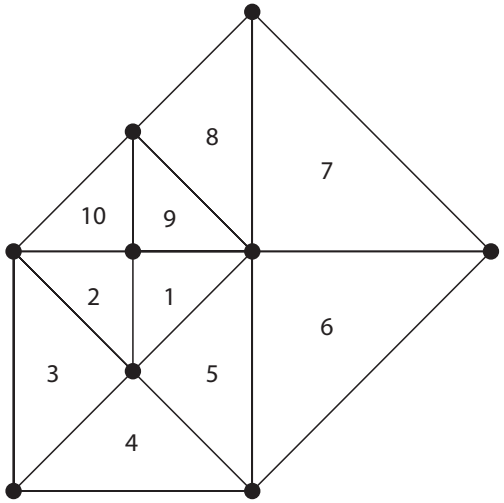
Mesh refinement: Longest Edge Bisection

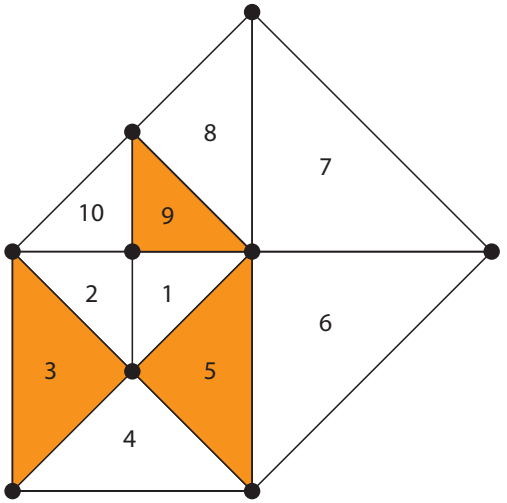


$\mathcal{M} = \{3, 5, 9\}$

$\mathcal{R} = \mathcal{M} \cup \{6, 7, 8\}$

Mesh refinement: Longest Edge Bisection

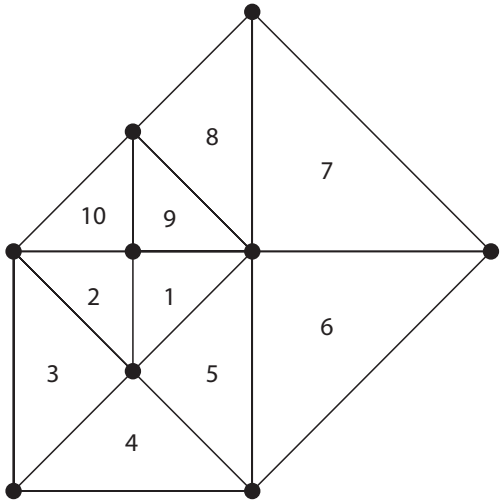


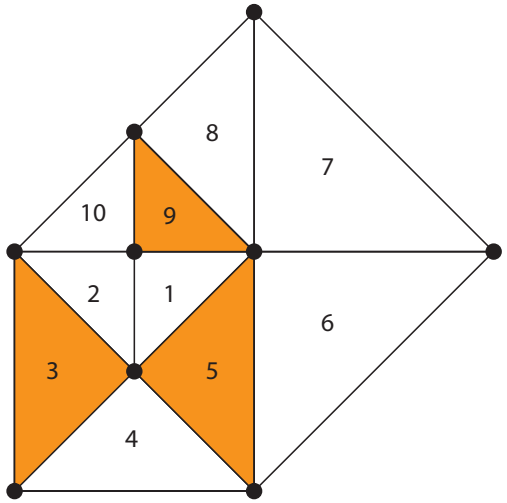


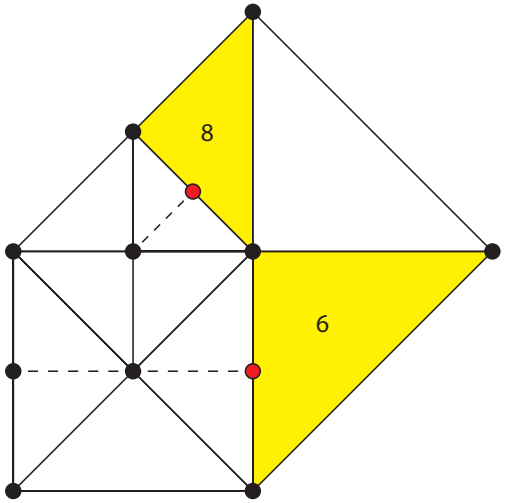
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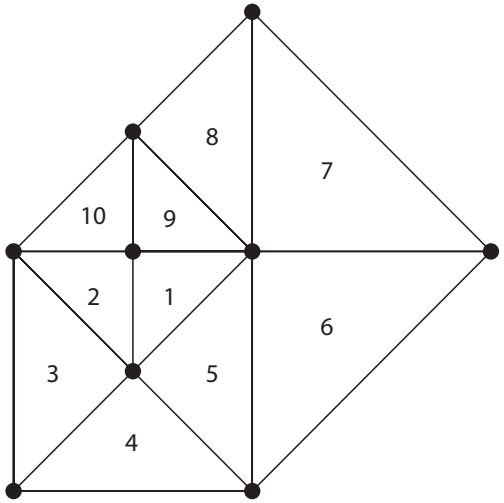


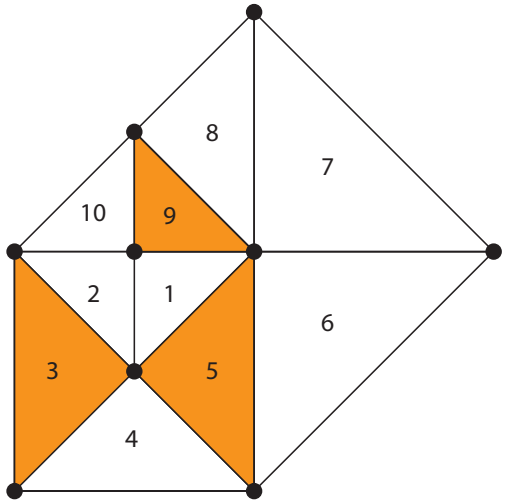


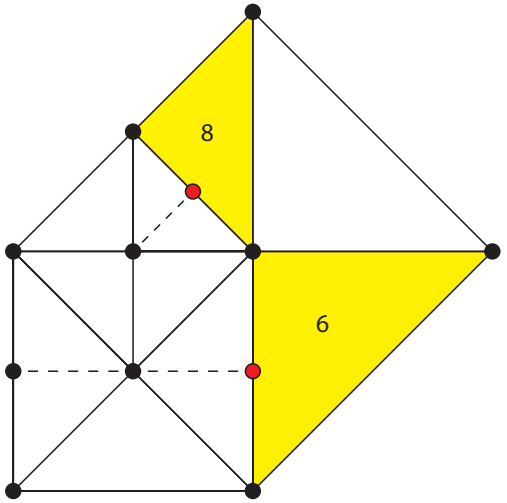
$\mathcal{M} = \{3, 5, 9\}$

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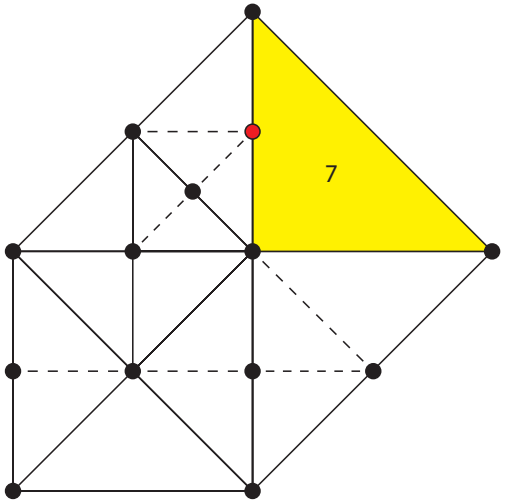
Mesh refinement: Longest Edge Bisection





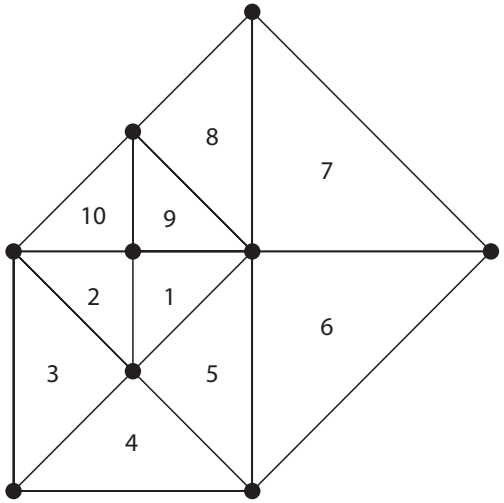


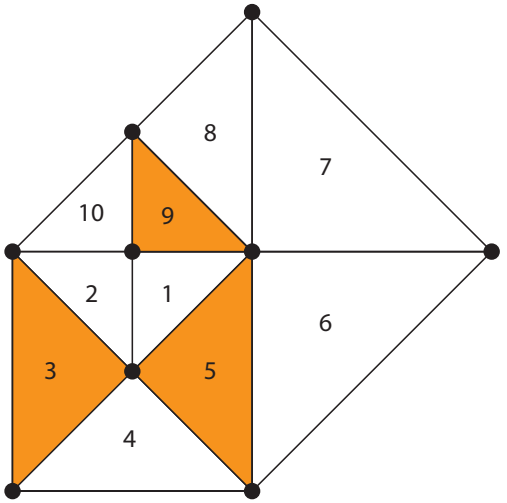
$\mathcal{M} = \{3, 5, 9\}$

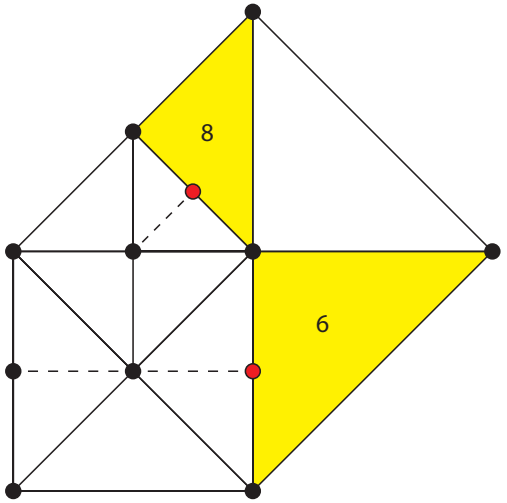


$\mathcal{R} = \mathcal{M} \cup \{6, 7, 8\}$

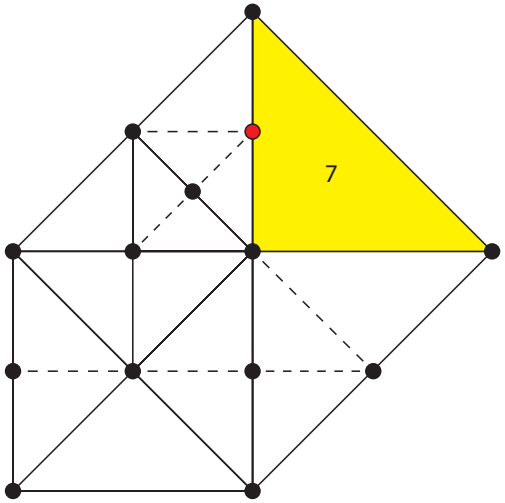
Mesh refinement: Longest Edge Bisection

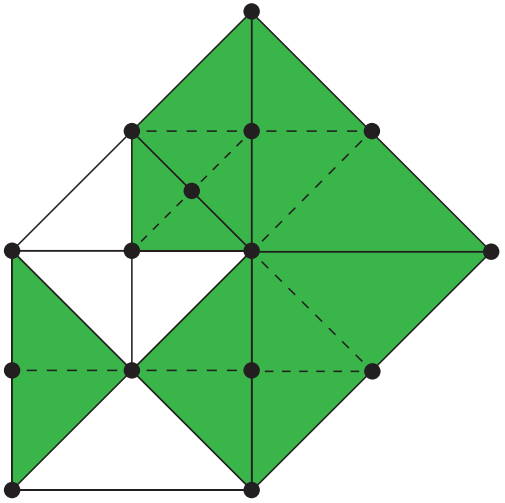






$\mathcal{M} = \{3, 5, 9\}$





$\mathcal{R} = \mathcal{M} \cup \{6, 7, 8\}$

Discretisation: linear system

Discrete weak formulation and discrete solution:

$$B(u_{X\mathcal{P}}, v) = F(v) \quad \forall\, v \in V_{X\mathcal{P}} \qquad u_{X\mathcal{P}}(\mathbf{x}, \mathbf{y}) = \sum_{\nu \in \mathcal{P}} \sum_{j=1}^{J} u_{\nu j}\, \varphi_j(\mathbf{x})\, P_\nu(\mathbf{y}).$$

The coefficients $\mathbf{u} = (u_{\nu j}) \in \mathbb{R}^{JP}$ can be found by solving the following $JP \times JP$ linear system

$$\mathbf{A}\,\mathbf{u} = \mathbf{f}, \quad \text{with} \quad \mathbf{A} := G_0 \otimes K_0 + \sum_{m=1}^{M} G_m \otimes K_m, \quad \mathbf{f} := g_0 \otimes f_0,$$

where

$$\text{(Stiffness matrices)} \qquad (K_m)_{ij} := \int_D a_m(\mathbf{x}) \nabla \varphi_j(\mathbf{x}) \cdot \nabla \varphi_i(\mathbf{x}) d\mathbf{x}, \quad \forall\, m \geq 0,$$

$$\text{(“Stochastic” matrices)} \quad \begin{cases} (G_0)_{\nu\mu} := \displaystyle\int_\Gamma P_\nu(\mathbf{y}) P_\mu(\mathbf{y}) d\pi(\mathbf{y}), & \forall\, \nu, \mu \in \mathcal{P}, \\ (G_m)_{\nu\mu} := \displaystyle\int_\Gamma y_m P_\nu(\mathbf{y}) P_\mu(\mathbf{y}) d\pi(\mathbf{y}), & \forall\, \nu, \mu \in \mathcal{P},\ m \geq 1, \end{cases}$$

$$(g_0)_\mu := \int_\Gamma P_\mu(\mathbf{y}) d\pi(\mathbf{y}), \qquad \text{and} \qquad (f_0)_i := \int_D f(\mathbf{x}) \varphi_i(\mathbf{x}) d\mathbf{x}.$$

Sparsity pattern of A

▷ $K_m \in \mathbb{R}^{J\times J}$, $G_m \in \mathbb{R}^{P\times P}$, $\mathbf{A} \in \mathbb{R}^{JP\times JP}$.

Example: let $\mathcal{P}_\mathcal{P}$ be a space of "complete polynomials", i.e., (P_ν) are polynomials of total degree n in the first M dimensions. Then

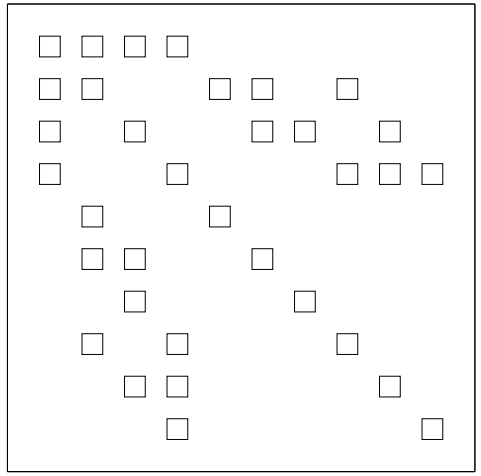
$$P = \dim(\mathcal{P}_\mathcal{P}) = \frac{(n+M)!}{n!\,M!}$$

On the right, each square represents a $J \times J$ matrix.

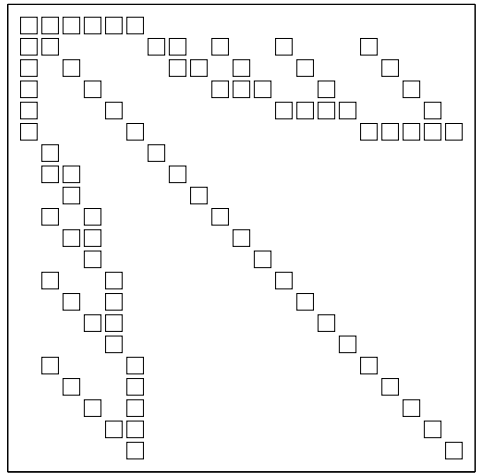
▷ Observe how ordering the indices $\nu \in \mathcal{P}$ in different ways changes the sparsity pattern of $\mathbf{A}$.

▷ Requires optimal solvers and preconditioning...

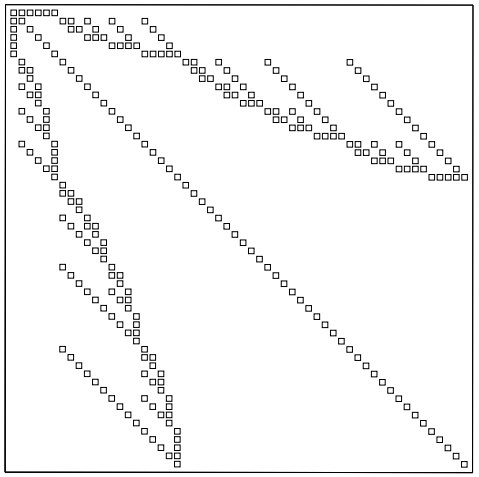
$M = 3,\ n = 2$



$M = 5,\ n = 2$



$M = 5,\ n = 3$



$M = 5,\ n = 3$

