We have $V\hat{x}\rho = V_{x\rho} + \sum_{\nu}\sum_{j} V_{j\nu} P_{\nu}$ ⟶

$$\|V\hat{x}\rho\|^2_{B_0} = B_0(V\hat{x}\rho, V\hat{x}\rho) = B_0(V\hat{x}\rho, V_{x\rho} + \Sigma\Sigma)$$

$$= B_0(V\hat{x}\rho, V_{x\rho}) + \sum_{\nu}\sum_{j} B_0(V\hat{x}\rho, V_{j\nu}P_{\nu})$$

(6.20) - (6.21)

$$= B_0(g_{x\rho}V\hat{x}\rho, V_{x\rho}) + \sum_{\nu}\sum_{j} B_0(g_{j\nu}V\hat{x}\rho, V_{j\nu}P_{\nu})$$

(CS)

$$\leq \|g_{x\rho}V\hat{x}\rho\|_{B_0}\|V_{x\rho}\|_{B_0} + \sum_{\nu}\sum_{j}\|g_{j\nu}V\hat{x}\rho\|_{B_0}\|V_{j\nu}P_{\nu}\|_{B_0} \qquad (1)$$

Now use CS for sums which reads as

$$\sum_{k=1} a_k b_k \leq \left(\sum_{k=1} a_k^2\right)^{1/2}\left(\sum_{k=1} b_k^2\right)^{1/2} \qquad (2)$$

hence image that (1) is simply

$$= \underbrace{\|g_{x\rho}V\hat{x}\rho\|_{B_0}}_{=a_1}\underbrace{\|V_{x\rho}\|_{B_0}}_{b_1} + \underbrace{\sum_{\nu}\sum_{j}}_{\Sigma_{k=2}}\underbrace{\|g_{j\nu}V\hat{x}\rho\|_{B_0}}_{=a_k}\underbrace{\|V_{j\nu}P_{\nu}\|_{B_0}}_{=b_k}$$

(2)

$$\leq \left(\sum_{k=1} a_k^2\right)^{1/2}\left(\sum_{k=1} b_k^2\right)^{1/2}$$

$$= \underbrace{\left(a_1^2 + \sum_{k=2} a_k^2\right)^{1/2}}\left(b_1^2 + \sum_{k=2} b_k^2\right)^{1/2}$$

$$= \left(\|g_{x\rho}V\hat{x}\rho\|^2_{B_0} + \sum_{\nu}\sum_{j}\|g_{j\nu}V\hat{x}\rho\|^2_{B_0}\right)^{1/2}\left(\|V_{x\rho}\|^2_{B_0} + \sum_{\nu}\sum_{j}\|V_{j\nu}P_{\nu}\|^2_{B_0}\right)^{1/2}$$

which is the first inequality of STEP 1 in LEMMA 6.4

LEMMA 6.2

$$\| a_0^{1/2} \nabla\Big(\sum_{j=1}^{N_y} w_j\Big)\|^2_{L^2(T)} = \int_T a_0(x)\, \underbrace{\nabla\Big(\sum_j^{N_y} w_j\Big)\cdot\nabla\Big(\sum_i^{N_y} w_i\Big)}_{\|}\, dx$$

$$= \int_T a_0(x)\Big[\sum_{j=1}^{N_y} \nabla w_j\cdot\nabla w_j + 2\sum_{j\neq i}\nabla w_j\cdot\nabla w_i\Big]\,dx$$

$$= \sum_{j=1}^{N_y}\int_T a_0(x)\nabla w_j\cdot\nabla w_j + 2\sum_{j\neq i}\underbrace{\int_T a_0(x)\nabla w_j\cdot\nabla w_i\,dx}_{= a_0^{1/2}\nabla w_j\cdot a_0^{1/2}\nabla w_i}$$

$$\overset{(CS)}{\leq} \sum_{j=1}^{N_y}\| a_0^{1/2}\nabla w_j\|^2_{L^2(T)} + 2\sum_{j\neq i}\Big(\int_T a_0\nabla w_j^2\Big)^{1/2}\Big(\int_T a_0\nabla w_i^2\Big)^{1/2}$$

$$= \sum_{j}^{N_y}\| a_0^{1/2}\nabla w_j\|^2_{L^2(T)} + 2\sum_{j\neq i}\| a_0^{1/2}\nabla w_j\|_{L^2(T)}\,\| a_0^{1/2}\nabla w_i\|_{L^2(T)}$$

$$\equiv \Big(\sum_{j=1}^{N_y}\| a_0^{1/2}\nabla w_j\|_{L^2(T)}\Big)^2$$

$$\leq K\sum_{j=1}^{N_y}\| a_0^{1/2}\nabla w_j\|^2_{L^2(T)} \qquad (1)$$

↑ dovrebbe essere perché for all w_j associated with midpoints j not in $T \Rightarrow \nabla w_j = 0$ since $w_j = 0$ on T

⇓

$$\| a_0^{1/2}\nabla\Big(\sum_j^{N_y} w_j\Big)\|_{L^2(T)} \leq \sum_{j=1}^{N_y}\| a_0^{1/2}\nabla w_j\|_{L^2(T)} \leq \sqrt{K}\Big(\sum_j^{N_y}\| a_0^{1/2}\nabla w_j\|^2_{L^2(T)}\Big)^{1/2} \qquad (2)$$

Then:

$$\| a_0^{1/2}\nabla\Big(\sum_j^{N_y} w_j\Big)\|^2_{L^2(D)} \equiv \int_D a_0(x)\nabla\Big(\sum w_i\Big)\cdot\nabla\Big(\sum w_j\Big)\,dx$$

$$\equiv \sum_{T\in\mathcal{T}}\underbrace{\int_T a_0(x)\nabla\Big(\sum w_i\Big)\cdot\nabla\Big(\sum w_j\Big)\,dx}_{\equiv\ \| a_0^{1/2}\nabla\left(\sum w_i\right)\|^2_{L^2(T)}}$$

$$\overset{(1)}{\leq} \sum_{T\in\mathcal{T}} K\sum_{j=1}^{N_y}\underbrace{\| a_0^{1/2}\nabla w_j\|^2_{L^2(T)}}$$

$$= K\sum_{j=1}^{N_y}\underbrace{\sum_{T\in\mathcal{T}}\int_T a_0^{1/2}\nabla w_j\cdot\nabla w_j\,dx}_{\equiv\ K\sum_{j=1}^{N_y}\| a_0^{1/2}\nabla w_j\|^2_{L^2(D)}} \qquad (3)$$

which concludes STEP 1

$$\| g_{x\rho} V_{\hat{x}\rho} + \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \|_{B_0}^2 = B_0 \left(g_{x\rho} V_{\hat{x}\rho} + \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \, , \, g_{x\rho} V_{\hat{x}\rho} + \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \right)$$

$$= \| g_{x\rho} V_{\hat{x}\rho} \|_{B_0}^2 + \underbrace{2 B_0 \left(g_{x\rho} V_{\hat{x}\rho} , \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \right)} + \| \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \|_{B_0}^2$$

Since $(a-b)^2 = a^2 + b^2 - 2ab \geq 0 \Rightarrow 2ab \leq a^2 + b^2$

$$\Downarrow$$

$$\leq \| g_{x\rho} V_{\hat{x}\rho} \|_{B_0}^2 + \overbrace{\| g_{x\rho} V_{\hat{x}\rho} \|_{B_0}^2 + \| \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \|_{B_0}^2} + \| \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \|_{B_0}^2$$

$$= 2 \| g_{x\rho} V_{\hat{x}\rho} \|_{B_0}^2 + 2 \| \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \|_{B_0}^2$$

$$= 2 \left(\| g_{x\rho} V_{\hat{x}\rho} \|_{B_0}^2 + \| \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \|_{B_0}^2 \right)$$

$$\Downarrow$$

$$\| g_{x\rho} V_{\hat{x}\rho} + \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \|_{B_0} \leq \sqrt{2} \left(\| g_{x\rho} V_{\hat{x}\rho} \|_{B_0}^2 + \| \sum_{\nu} \sum_{j} g_{j\nu} V_{\hat{x}\rho} \|_{B_0}^2 \right)^{1/2}$$

which is (6.25) of STEP 2 of LEMMA 6.4