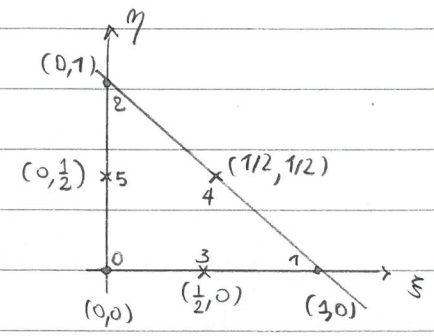
⇨ Basis Functions on the Reference Triangular Element — CASE $\mathbb{P}_2$

Any quadratic function $u \in \mathbb{P}_2(\hat{K})$ is of the form

$$u(\xi,\eta) = c_0 + c_1\xi + c_2\eta + c_3\xi^2 + c_4\xi\eta + c_5\eta^2 \qquad (1)$$



d.o.f = 6

Imposing the values of the nodes:

$$\begin{cases} U_0 = u(0,0) = c_0 \\ U_1 = u(1,0) = c_0 + c_1 + c_3 \\ U_2 = u(0,1) = c_0 + c_2 + c_5 \\ U_3 = u(1/2,0) = c_0 + \frac{1}{2}c_1 + \frac{1}{4}c_3 \\ U_4 = u(1/2,1/2) = c_0 + \frac{1}{2}c_1 + \frac{1}{2}c_2 + \frac{1}{4}c_3 + \frac{1}{4}c_4 + \frac{1}{4}c_5 \\ U_5 = u(0,1/2) = c_0 + \frac{1}{2}c_2 + \frac{1}{4}c_5 \end{cases}$$

that is a linear system $U = Ac$. By inverting the matrix A, we can obtain the coefficients $\Rightarrow c = A^{-1}U$. We get

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 4 & 0 & 0 \\ -3 & 0 & -1 & 0 & 0 & 4 \\ 2 & 2 & 0 & -4 & 0 & 0 \\ 4 & 0 & 0 & -4 & 4 & -4 \\ 2 & 0 & 2 & 0 & 0 & -4 \end{pmatrix}}_{A^{-1}} \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} U_0 \\ -3U_0 - U_1 + 4U_3 \\ -3U_0 - U_2 + 4U_5 \\ 2U_0 + 2U_1 - 4U_3 \\ 4U_0 - 4U_3 + 4U_4 - 4U_5 \\ 2U_0 + 2U_2 - 4U_5 \end{pmatrix} \qquad (2)$$

Inserting the values of the e_i, $i=0\dots5$ in (1) and ordering the terms, we get:

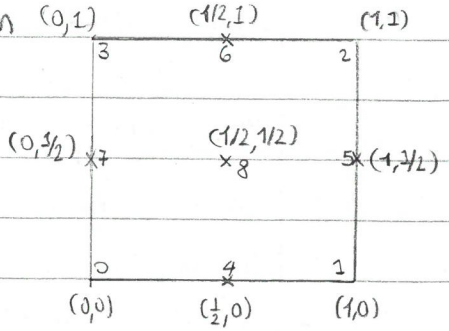
$$u(\xi,\eta) = U_0\hat{\varphi}_0 + U_1\hat{\varphi}_1 + U_2\hat{\varphi}_2 + U_3\hat{\varphi}_3 + U_4\hat{\varphi}_4 + U_5\hat{\varphi}_5 = \sum_{i=0}^{5} U_i\hat{\varphi}_i(\xi,\eta)$$

$\hat{\varphi}_0(\xi,\eta) = (1-\xi-\eta)(1-2\xi-2\eta)$	$\hat{\varphi}_3(\xi,\eta) = 4\xi(1-\xi-\eta)$
$\hat{\varphi}_1(\xi,\eta) = \xi(2\xi-1)$	$\hat{\varphi}_4(\xi,\eta) = 4\xi\eta$
$\hat{\varphi}_2(\xi,\eta) = \eta(2\eta-1)$	$\hat{\varphi}_5(\xi,\eta) = 4\eta(1-\xi-\eta)$

⇒ Basis Functions on the reference Quadrilateral Element — CASE $\mathbb{P}_2$

Any quadratic function $v \in \mathbb{P}_2(\hat{K})$ is written in the form

$$v(\xi,\eta) = c_0 + c_1\xi + c_2\eta + c_3\xi^2 + c_4\xi\eta + c_5\eta^2 + c_6\xi^2\eta + c_7\xi\eta^2 + c_8\xi^2\eta^2 \quad (1)$$



d.o.f = 9

Imposing the values of the nodes:

$$\begin{cases}
v_0 = v(0,0) = c_0 \\
v_1 = v(1,0) = c_0 + c_1 + c_3 \\
v_2 = v(1,1) = c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 \\
v_3 = v(0,1) = c_0 + c_2 + c_5 \\
v_4 = v(1/2,0) = c_0 + \tfrac{1}{2}c_1 + \tfrac{1}{4}c_3 \\
v_5 = v(1,1/2) = c_0 + c_1 + \tfrac{1}{2}c_2 + c_3 + \tfrac{1}{2}c_4 + \tfrac{1}{4}c_5 + \tfrac{1}{2}c_6 + \tfrac{1}{4}c_7 + \tfrac{1}{4}c_8 \\
v_6 = v(1/2,1) = c_0 + \tfrac{1}{2}c_1 + c_2 + \tfrac{1}{4}c_3 + \tfrac{1}{2}c_4 + c_5 + \tfrac{1}{4}c_6 + \tfrac{1}{2}c_7 + \tfrac{1}{4}c_8 \\
v_7 = v(0,1/2) = c_0 + \tfrac{1}{2}c_2 + \tfrac{1}{4}c_5 \\
v_8 = v(1/2,1/2) = c_0 + \tfrac{1}{2}c_1 + \tfrac{1}{2}c_2 + \tfrac{1}{4}c_3 + \tfrac{1}{4}c_4 + \tfrac{1}{4}c_5 + \tfrac{1}{8}c_6 + \tfrac{1}{8}c_7 + \tfrac{1}{16}c_8
\end{cases}$$

Again we obtain $V = AC$ and we get the coefficients c_i, $i = 0 \dots 8$ by inverting A.
$\Rightarrow C = A^{-1}V$. The Matrix A^{-1} is

$$A^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -1 & 0 & 0 & 4 & 0 & 0 & 0 \\
-3 & 0 & 0 & -1 & 0 & 0 & 4 & 0 \\
2 & 2 & 0 & 0 & -4 & 0 & 0 & 0 \\
9 & 3 & 1 & 3 & -12 & -4 & -12 & 16 \\
2 & 0 & 0 & 2 & 0 & 0 & -4 & 0 \\
-6 & -6 & -2 & -2 & 12 & 8 & 8 & -16 \\
-6 & -2 & -2 & -6 & 8 & 4 & 12 & -16 \\
4 & 4 & 4 & 4 & -8 & -8 & -8 & 16
\end{pmatrix}$$

so that we get the following coefficients:

$$
\begin{cases}
c_0 = u_0 \\
c_1 = -3u_0 - u_1 + 4u_4 \\
c_2 = -3u_0 - u_3 + 4u_7 \\
c_3 = -4u_4 + 2u_1 + 2u_0 \\
c_4 = 9u_0 + 3u_1 + u_2 + 3u_3 - 12u_4 - 4u_5 - 4u_6 - 12u_7 + 16u_8 \\
c_5 = 2u_0 + 2u_3 - 4u_7 \\
c_6 = 12u_4 - 6u_1 - 2u_2 - 2u_3 - 6u_0 + 8u_5 + 4u_6 + 8u_7 - 16u_8 \\
c_7 = -6u_0 - 2u_1 - 2u_2 - 6u_3 + 8u_4 + 4u_5 + 8u_6 + 12u_7 - 16u_8 \\
c_8 = 4u_0 + 4u_1 + 4u_2 + 4u_3 - 8u_4 - 8u_5 - 8u_6 - 8u_7 + 16u_8
\end{cases}
$$

By inserting these coefficients in (1) and after (a lot!) of calculations to order the terms we finally get:

$$u(\xi,\eta) = \sum_{i=0}^{8} u_i \hat{\varphi}_i(\xi,\eta)$$

$$\hat{\varphi}_0(\xi,\eta) = (1-3\xi+2\xi^2)(1-3\eta+2\eta^2) \qquad \hat{\varphi}_2(\xi,\eta) = (\xi-2\xi^2)(\eta-2\eta^2)$$

$$\hat{\varphi}_1(\xi,\eta) = (1-3\xi+2\xi^2)(2\xi^2-\xi) \qquad \hat{\varphi}_3(\xi,\eta) = (1-3\xi+2\xi^2)(2\eta^2-\eta)$$

$$\hat{\varphi}_4(\xi,\eta) = (4-12\eta-8\eta^2)(\xi-\xi^2) \qquad \hat{\varphi}_6(\xi,\eta) = 4(\eta-2\eta^2)(\xi^2-\xi)$$

$$\hat{\varphi}_5(\xi,\eta) = 4(2\xi^2-\xi)(\eta-\eta^2) \qquad \hat{\varphi}_7(\xi,\eta) = (4-12\xi+8\xi^2)(\eta-\eta^2)$$

$$\hat{\varphi}_8(\xi,\eta) = 16(\xi-\xi^2)(\eta-\eta^2)$$

See Matlab code for the plots...