Eigel coefficients

$$a(x,y) = 1 + \sum_{m=1}^{\infty} \underbrace{A m^{-\sigma} \cos(2\pi\beta_1(m)x_1)\cos(2\pi\beta_2(m)x_2)}_{=: a_m(x)} \gamma_m \qquad (1)$$

$$\beta_1(m) := m - \frac{k(m)(k(m)+1)}{2}$$

$$\beta_2(m) := k(m) - \beta_1(m)$$

$$k(m) := \left\lfloor -\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + 2m} \right\rfloor$$

$$\nabla a_m = \left(\partial a_m/\partial x_1 \,,\, \partial a_m/\partial x_2 \right)$$

First values of β_1, β_2: (2)

m	β_1	β_2
1	0	1
2	1	0
3	0	2
4	1	1
5	2	0
6	0	3
7	1	2

$$\frac{\partial a_m}{\partial x_1} = -2\pi\beta_1(m) A m^{-\sigma} \sin(2\pi\beta_1(m)x_1)\cos(2\pi\beta_2(m)x_2)$$

$$\frac{\partial a_m}{\partial x_2} = -2\pi\beta_2(m) A m^{-\sigma} \cos(2\pi\beta_1(m)x_1)\sin(2\pi\beta_2(m)x_2) \qquad (3)$$

Assumption:

$$\max_{T\in\mathcal{T}_R} \left(h_T \, \| (\nabla a_m)/a_0 \|_{L^\infty(T)} \right) \le C_{\text{supp}} \, \| a_m/a_0 \|_{L^\infty(D)} \qquad (4)$$

RHS of (4):

$$\| a_m/a_0 \|_{L^\infty(D)} \equiv \| a_m \|_{L^\infty(D)} = A m^{-\sigma} \underbrace{\max_{(x_1,x_2)\in D} \left| \cos(2\pi\beta_1(m)x_1)\cos(2\pi\beta_2(m)x_2) \right|}$$

this max is 1, and it is reached when (x_1,x_2) is s.t. both cosines are ± 1

There we have:

$$\begin{cases} \cos(2\pi\beta_1(m)x_1) = \pm 1 \iff \beta_1 = 0 \text{ OR } 2\pi\beta_1(m)x_1 = \ell\pi \Rightarrow x_1 = \dfrac{\ell}{2\beta_1(m)}, \ \ell\in\mathbb{Z} \\ \cos(2\pi\beta_2(m)x_2) = \pm 1 \iff \beta_2 = 0 \text{ OR } 2\pi\beta_2(m)x_2 = m\pi \Rightarrow x_2 = \dfrac{m}{2\beta_2(m)}, \ m\in\mathbb{Z} \end{cases}$$

Hence cases are:

1) $\beta_1 = 0$ AND $x_2 = \dfrac{m}{2\beta_2(m)}$ (whatever $x_1 \in [-1,1]$), $(\beta_2 \neq 0)$

2) $x_1 = \dfrac{\ell}{2\beta_1(m)}$ AND $\beta_2 = 0$ (" $x_2 \in [-1,1]$), $(\beta_1 \neq 0)$

3) $x_1 = \dfrac{\ell}{2\beta_1(m)}$ AND $x_2 = \dfrac{m}{2\beta_2(m)}$ $(\beta_1 \neq 0, \beta_2 \neq 0)$

do such points (x_1,x_2) exist in the domain?

In all 3 cases, we should ensure that exists such a point $(x_1, x_2) \in D$ (this is square, L-shaped). That is:

case 1) $x_2 = \frac{n}{2\beta_2(m)} \in [-1, 1]$ with $\beta_2 \neq 0$, $n \in \mathbb{Z}$, $m \in \mathbb{N}$

- $x_2 \leq 1 \Rightarrow n \leq 2\beta_2(m)$: does exist an $n \in \mathbb{Z}$ s.t. $n \in 2\beta_2(m)\ \forall m$? Yes! Lower bound of β_2 (without $=0$) is $\beta_2 = 1$ and the it has increasing values. Then $m = 0, 1$, for example satisfy the condition
- $x_2 \geq -1 \Rightarrow -2\beta_2(m) \leq n$

  In the same way, $-\beta_2$ is at the most $=1$ (without 0) and then it decreases. Hence, again, $n = 0, 1$, for example, satisfy the condition.

case 2) Argue as above

case 3) Argue as above, again: e.g. when $m = 4 \Rightarrow \beta_1 = \beta_2 = 1$, then:

$x_1 = \ell/2$, $x_2 = n/2$, $\ell, n \in \mathbb{Z}$

In conclusion:

$$\| a_m \setminus a_0 \|_{L^\infty(D)} \equiv A\, m^{-\sigma} \qquad (5)$$

LHS of (4)

Let us consider first the following

$$\| (\nabla a_m) / a_0 \|_{L^\infty(D)} \equiv \| \nabla a_m \|_{L^\infty(D)} := \max_{(x_1,x_2)\in D} \underbrace{\sqrt{\left|\frac{\partial a_m}{\partial x_1}\right|^2 + \left|\frac{\partial a_m}{\partial x_2}\right|^2}}_{|\nabla a_m|}$$

$$= \max_D \left(4\pi^2 (A m^{-\sigma})^2 \left[\beta_1^2 \left(\sin(2\pi\beta_1 x_1)\cos(2\pi\beta_2 x_2)\right)^2 + \beta_2^2 \left(\cos(2\pi\beta_1 x_1)\sin(2\pi\beta_2 x_2)\right)^2 \right] \right)^{1/2}$$

$$\equiv 2\pi A m^{-\sigma} \underbrace{\max_D}_{\hookrightarrow} \underbrace{\left(\beta_1^2 \left(\sin(2\pi\beta_1 x_1)\cos(2\pi\beta_2 x_2)\right)^2 + \beta_2^2 \left(\cos(2\pi\beta_1 x_1)\sin(2\pi\beta_2 x_2)\right)^2 \right)^{1/2}}_{\circledast} \qquad (6)$$

↳ We can distinguish here 3 cases:

case 1) $\beta_1(m) = 0$ and $\beta_2 \neq 0$. This gives

$$\circledast = \beta_2 \max_D \left| \underbrace{\cos(0)}_{\equiv 1} \sin(2\pi\beta_2 x_2) \right| \equiv \beta_2 \max_D \left| \sin(2\pi\beta_2 x_2) \right| = \beta_2$$

since it will exists a coordinate x_2 s.t. $\sin(2\pi\beta_2 x_2) \equiv \pm 1 \Rightarrow 2\pi\beta_2 x_2 = m\frac{\pi}{2}$

$\Rightarrow x_2 = \frac{m}{4\beta_2} \quad m \in \mathbb{Z}\setminus\{0\}$.

case 2) $\beta_1(m) \neq 0$ and $\beta_2 = 0$. Then

$$\circledast = \beta_1 \max_D \left| \sin(2\pi\beta_1 x_1) \underbrace{\cos(0)}_{=1} \right| \equiv \beta_1 \max_D \underbrace{|\sin(2\pi\beta_1 x_1)|}_{=1} = \beta_1$$

case 3) $\beta_1 \neq 0$ and $\beta_2 \neq 0$. Then it's clear that that max of $\circledast$ is achieved when $\underbrace{\sin(2\pi\beta_1 x_1)\cos(2\pi\beta_2 x_2)}_{=:A} = \pm 1$ and $\underbrace{\cos(2\pi\beta_1 x_1)\sin(2\pi\beta_2 x_2)}_{=:B} = \pm 1$.

Indeed it happens that points $(x_1 x_2)$ which give $A = \pm 1$ will in turn give $B = 0$. Obvious. To see this consider:

$$A = \pm 1 \Rightarrow \begin{cases} \sin(2\pi\beta_1 x_1) = \pm 1 \Rightarrow 2\pi\beta_1 x_1 = \ell\pi/2 \Rightarrow x_1 = \ell/4\beta_1, & \ell \in \mathbb{Z}\setminus\{0\} \\ \cos(2\pi\beta_2 x_2) = \pm 1 \Rightarrow 2\pi\beta_2 x_2 = m\pi \Rightarrow x_2 = m/2\beta_2, & m \in \mathbb{Z} \end{cases}$$

and now put these x_1, x_2 into B:

$$B = \cos\left(2\pi\beta_1 \frac{\ell}{4\beta_1}\right) \sin\left(2\pi\beta_2 \frac{m}{2\beta_2}\right) = \underbrace{\cos\left(\ell\frac{\pi}{2}\right)}_{=0} \underbrace{\sin(m\pi)}_{=0} = 0.$$

Therefore what happens im this case is

$$\circledast = \max_D \left(\beta_1^2 \cdot 1 + \beta_2 \cdot 0\right)^{1/2} = \beta_1$$

Similarly, looking for the values $(x_1 x_2)$ s.t. $B = 1$ will give $A = 0$ and then

$$\circledast = \max_D \left(\beta_1^2 \cdot 0 + \beta_2^2 \cdot 1\right)^{1/2} = \beta_2.$$

In conclusion, in any case you get $\circledast$ either equal to β_1 or β_2, and this is resumed as the $\max\{\beta_1(m), \beta_2(m)\}$.

Then, coming back to (6), we have:

$$\|(\nabla a_m)/a_0\|_{L^\infty(D)} \equiv 2\pi A m^{-\sigma} \max\{\beta_1(m), \beta_2(m)\} \qquad (7)$$

Now, for a $T \in \mathcal{T}$, we have

$$\Rightarrow h_T \|(\nabla a_m)/a_0\|_{L^\infty(T)} \leq h_T \|(\nabla a_m)/a_0\|_{L^\infty(D)}$$

and picking the max contribution on both sides of the above inequality

$$\max_{T\in\mathcal{T}} \left(h_T \, \| (\nabla a_m)/a_0 \|_{L^\infty(T)} \right) \leq \max_{T\in\mathcal{T}} \left(h_T \, \| (\nabla a_m)/a_0 \|_{L^\infty(D)} \right)$$

$$= \| (\nabla a_m)/a_0 \|_{L^\infty(D)} \underbrace{\max_{T\in\mathcal{T}} h_T}_{=: \bar{h}}$$

$$\overset{(7)}{=} 2\pi A m^{-\sigma} \bar{h} \max\{\beta_1(m), \beta_2(m)\} \qquad (8)$$

Therefore, in principle the assumption (4) can be modified using (8):

$$\max_{T\in\mathcal{T}} \left(h_T \, \| \nabla a_m / a_0 \|_{L^\infty(T)} \right) \leq \underbrace{\bar{h} \, \| \nabla a_m / a_0 \|_{L^\infty(D)}}_{\| \,=\, 2\pi A m^{-\sigma} \bar{h} \max\{\beta_1, \beta_2\}} \leq C_{supp} \, \| a_m / a_0 \|_{L^\infty(D)} \qquad (9)$$

Anyway, the code computes effectively the LHS and the RHS of (8) for any coefficient. In the case of Fourier coefficients, the LHS can be "compared" with the middle analytical term