$$\|\nabla a_m / a_0\|_{L^\infty(T)} = \|\nabla a_m\|_{L^\infty(T)} = \max_{x \in T} \sqrt{\left|\frac{\partial a_m}{\partial x_1}\right|^2 + \left|\frac{\partial a_m}{\partial x_2}\right|^2}$$

$\|a_0\|_{L^\infty(T)} = 1$

$$a(x,\theta) = 1 + \sum_{m=1}^{\infty} \underbrace{A m^{-\sigma} \cos(2\pi\beta_1(m) x_1) \cos(2\pi\beta_2(m) x_2)}_{= a_m} y_m$$

$$\frac{\partial a_m}{\partial x_1} = -2\pi\beta_1(m) A m^{-\sigma} \sin(2\pi\beta_1(m) x_1) \cos(2\pi\beta_2(m) x_2)$$

$$\frac{\partial a_m}{\partial x_2} = -2\pi\beta_2(m) A m^{-\sigma} \cos(2\pi\beta_1(m) x_1) \sin(2\pi\beta_2(m) x_2)$$

$$\left|\frac{\partial a_m}{\partial x_1}\right|^2 = 4\pi^2 \beta_1(m)^2 (A m^{-\sigma})^2 \left(\sin(2\pi\beta_1 x_1)\cos(2\pi\beta_2 x_2)\right)^2$$

$$\left|\frac{\partial a_m}{\partial x_2}\right|^2 = 4\pi^2 \beta_2(m)^2 (A m^{-\sigma})^2 \left(\cos(2\pi\beta_1(m) x_1)\sin(2\pi\beta_2 x_2)\right)^2$$

$$\Rightarrow \max_{(x_1,x_2)\in T} \sqrt{4\pi^2 (A m^{-\sigma})^2 \left[\beta_1(m)^2 (\sin()\cos())^2 + \beta_2(m)^2 (\cos()\sin())^2\right]} =$$

$$= 2\pi A m^{-\sigma} \underbrace{\max_{(x_1,x_2)\in T} \sqrt{\beta_1(m)^2 (\sin()\cos())^2 + \beta_2(m)^2 (\cos()\sin())^2}}_{\|}$$

$$= 2\pi A m^{-\sigma} \max\{\beta_1(m), \beta_2(m)\}$$

because when β_1 or $\beta_2 = 0$, then it remain something like $\max_{(x_1,x_2)} \sqrt{\beta_1^2 \cdot 1 + 0} = \beta_1$ or $\max_{(x_1,x_2)} \sqrt{0 + \beta_2^2 \cdot 1} = \beta_2$

If $\beta_1 \neq 0$, $\beta_2 \neq 0$ however there's a point (x_1,x_2) s.t. one component gives 0 and the other gives 1. Eg. $\beta_1 = \beta_2 = 1$ ($m=4$) then $(x_1,x_2) = (0, 1/4) \Rightarrow \sqrt{\beta_1^2(0) + \beta_2^2(\underbrace{\cos(0)\sin(\pi/2)}_{=1})} \equiv \beta_2$

Then

$$\max_{T\in T_R} \left(h_T \|\nabla a_m / a_0\|_{L^\infty(T)}\right) \equiv \max_{T\in T_R} \left(h_T 2\pi A m^{-\sigma} \max\{\beta_1(m), \beta_2(m)\}\right)$$

$$\equiv 2\pi A m^{-\sigma} \max\{\beta_1(m), \beta_2(m)\} \underbrace{\max_{T\in T_R}(h_T)}_{h_{MAX}}$$

$$\equiv h_{MAX}\, 2\pi A m^{-\sigma} \max\{\beta_1(m), \beta_2(m)\} \quad (*)$$

$\equiv$ LHS inf norm

Then

$$\max_{T\in T_R} \left(h_T \|\nabla a_m / a_0\|_{L^\infty(T)}\right) \leq C \underbrace{\|a_m / a_0\|_{L^\infty(\Omega)}}_{\text{pick } C=1} = A m^{-\sigma} \equiv \text{RHS inf norm}$$

$$\Rightarrow 2\pi\, h_{MAX} \max\{\beta_1(m), \beta_2(m)\} \leq C \equiv 1$$

L-Shape, $-1 \le x_1 \le 1$, $-1 \le x_2 \le 1$



	β_1	β_2	
$m=1$	0	1	$a_1(x_1,x_2) = A(1)^{-2} \underbrace{\cos(0)}_{=1} \cos(2\pi x_2) = A \cdot \cos(2\pi x_2)$

$$\Rightarrow \|a_1/a_0\|_{L^\infty(D)} = A \underbrace{\max_{x\in D} |\cos(2\pi x_2)|}_{=1} = A$$

$m=2$	1	0	$a_2(x_1,x_2) = A(2)^{-2} \cos(2\pi x_1) \underbrace{\cos(0)}_{=1} \equiv$

$$\Rightarrow \|a_2/a_0\|_{L^\infty(D)} = A(1)^{-2} \underbrace{\max_{x\in D} |\cos(2\pi x_1)|}_{} = A(2)^{-2}$$

$m=3$	0	2	$\Rightarrow a_3(x_1,x_2) = A(3)^{-2} \underbrace{\cos(0)}_{} \cos(2\pi \cdot 2 x_2)$

$$\Rightarrow \|a_3/a_0\|_{L^\infty(D)} = A(3)^{-2} \max_{x\in D} |\cos(4\pi x_2)| = A(3)^{-2}$$

(*) $\|a_m/a_0\|_{L^\infty(D)} = A m^{-\sigma} \max_{x\in D} |\cos(2\pi\beta_1 x_1)\cos(2\pi\beta_2 x_2)|$

$\equiv A m^{-\sigma}$

$\cos(\underbrace{2\pi\beta_1(m)}_{k\pi} x_1) \equiv \pm 1 \iff 2\pi\beta_1(m) x_1 \equiv k\pi \quad k=0,1,\dots$

$k\pi x_1$, k even, $k=0,2,4,6\dots$ $\quad 2\beta_1(m) x_1 \equiv m-1$

$$x_1 = \frac{m-1}{2\beta_1(m)} \qquad \beta_1(m) \neq 0$$

$$\boxed{\|a_m/a_0\|_{L^\infty(D)} = A m^{-\sigma}}$$

$\cos(2\pi\beta_2(m) x_2) \equiv \pm 1 \iff$ $\quad x_2 = \dfrac{m-1}{2\beta_2(m)} \quad \beta_2(m) \neq 0$

Anyway $(x_1,x_2) = (0,0) \in D \Rightarrow \cos(0)\cos(0) \equiv 1$ then (*) is certainly true for all $m \ge 1$ at $(0,0)$.

Otherwise, for example

even multiple of π

$\cos(\underbrace{2\pi\beta_1(m)}_{k=0,1,2} x_1) \equiv \cos(2k\pi x_1) = \pm 1 \iff 2k\pi x_1 = \ell\pi \quad \ell = -2,-1,0,1,2\dots \ (\ell \in \mathbb{Z})$

$\hookrightarrow x_1 = \dfrac{\ell}{2k}$, $k \in \mathbb{Z}\setminus\{0\}$ s.t. $x_1 \in [-1,1]$

The same for $\cos(2\pi\beta_2 x_2)$

t.g. $\ell=1$, $k=1$ $x_1 = 1/2$

$\ell=3$, $k=2$ $x_1 = 3/4$

$m=1$, $\beta_1=0$, $\beta_2=1$ $\Rightarrow \cos(0\cdot x_1)\cos(1\cdot 2\pi x_2) \Rightarrow (x_1,x_2) = \left(\frac{\ell_1}{2}, \frac{\ell_2}{2}\right) =$ $(0,0)$

Assumption

$$\max_{T \in \mathcal{T}_\ell} \left(h_T \, \| (\nabla a_m)/a_0 \|_{L^\infty(T)} \right) \leq C_{\text{supp}} \, \| a_m/a_0 \|_{L^\infty(D)}$$

$$\Downarrow$$

$$\max_{T \in \mathcal{T}_\ell} \left(h_T \, \| (\nabla a_m)/a_0 \|_{L^\infty(T)} \right) \leq \bar{h} \, \| (\nabla a_m)/a_0 \|_{L^\infty(D)} \leq C_{\text{supp}} \, \| a_m/a_0 \|_{L^\infty(D)}$$

$$\bar{h} := \max_{T \in \mathcal{T}_\ell} (h_T)$$

Eipel example : $a_0 = 1$, $a_m := A m^{-\sigma} \cos(2\pi \beta_1(m) x_1) \cos(2\pi \beta_2(m) x_2)$

$$\| (\nabla a_m)/a_0 \|_{L^\infty(D)} = 2\pi A m^{-\sigma} \max\{\beta_1(m), \beta_2(m)\}$$

$$\| a_m/a_0 \|_{L^\infty(D)} = A m^{-\sigma}$$

$$\hookrightarrow \max_{T \in \mathcal{T}_\ell} \left(h_T \, \| \nabla a_m/a_0 \|_{L^\infty(T)} \right) \leq 2\pi A m^{-\sigma} \bar{h} \max\{\beta_1, \beta_2\} \leq C_{\text{supp}} \, A m^{-\sigma}$$