$$a_m(x) = \alpha_m \cos(2\pi\beta_1(x)x_1)\cos(2\pi\beta_2(x)x_2), \quad m \geq 1 \qquad (1)$$

$$\alpha_m := A m^{-\sigma}, \quad m \geq 1 \qquad (2)$$

$$\alpha_{min} := \sum_{m=1}^{M} \alpha_m = \sum_{m=1}^{M} A m^{-\sigma} \qquad (3)$$

Eipel
Merdon
Neumann
field

$$a(x,y) := \frac{c}{\alpha_{min}} \left(\sum_{m=1}^{M} y_m a_m(x) + \alpha_{min} \right) + \varepsilon \equiv (c+\varepsilon) + \sum_{m=1}^{M} y_m \frac{a_m(x)\, c_1}{\alpha_{min}} \qquad (4)$$

then with $a_0(x) \equiv c + \varepsilon$ and $\overline{a_m}(x) = a_m(x)\frac{c}{\alpha_{min}}$ (5)

$$\hookrightarrow \quad a(x,y) = a_0(x) + \sum_{m=1}^{M} y_m \overline{a_m}(x) \qquad (6)$$

$$|a(x,y) - (c+\varepsilon)| \leq \sum_{m=1}^{M} \underbrace{|y_m|}_{\leq 1} \left| \frac{a_m(x)\, c}{\alpha_{min}} \right| \leq \frac{c}{\alpha_{min}} \sum_{m=1}^{M} |a_m(x)|$$

$$= \frac{c}{\alpha_{min}} \sum_{m=1}^{M} |A m^{-\sigma} \cos(\,)\cos(\,)|$$

$$\leq \frac{c}{\alpha_{min}} \underbrace{\sum_{m=1}^{M} A m^{-\sigma}}_{\equiv \alpha_{min}} \equiv c \qquad (7)$$

Then

$$0 < \varepsilon = -c + (c+\varepsilon) \leq a(x,y) \leq c + (c+\varepsilon) = 2c + \varepsilon \qquad (8)$$

the codomain of $a(x,y)$ is $[\varepsilon, 2c+\varepsilon]$. (9)

The field (6) has to satisfy

(I) $a_0^{min} \leq a_0(x) \leq a_0^{max}$

(II) $\tau := \frac{1}{a_0^{min}} \sum_{m=1}^{\infty} \| \overline{a_m} \|_{L^\infty(D)} < 1$

Then

(I) we have $a_0^{min} \equiv a_0^{max} \equiv c + \varepsilon$ ✓ (10)

(II) If we suppose "$M = \infty$" we will have

$$\tau = \frac{1}{(c+\varepsilon)} \sum_{m=1}^{\infty} \max_x \left| \frac{A m^{-\sigma} \cos(\,)\cos(\,)\, c}{\sum_{m=1}^{\infty} A m^{-\sigma}} \right| = \frac{1}{(c+\varepsilon)} \left(c \Big/ \sum_{m=1}^{\infty} A m^{-\sigma} \right) \sum_{m=1}^{\infty} A m^{-\sigma}$$

$$= \frac{c}{c+\varepsilon} < 1 \quad ✓ \qquad (11)$$

They pick:

$A = 0.6$
$\varepsilon = 5 \cdot 10^{-6}$
$c = 10^{-3}$
$\sigma = 2$

$\Rightarrow$

$a(x,y) \in [\varepsilon, 2c + \varepsilon]$

$a_0^{min} = a_0^{max} \equiv c + \varepsilon$

$\alpha_{min} = \sum_{m=1}^{\infty} A\, m^{-\sigma} = A\,\zeta(\sigma) = A\,\zeta(2) = A\frac{\pi^2}{6} \simeq 0.9869$

Mean Value and Variance

$\mathbb{E}[a(x,y)] = a_0(x) = c + \varepsilon$

$\mathrm{Var}[a(x,y)] = 1/3 \sum_{m=1}^{M} \overline{a_m(x)}^2$

$\frac{1}{2}\int_{-1}^{1} y_m^2\, dy = \frac{1}{2}\,\frac{y_m^3}{3}\Big|_{-1}^{1} = \frac{1}{2}\left(\frac{1}{3} - \frac{(-1)}{3}\right) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$

$\sum_{i=1}^{3} y_i^2 a_i^2 + 2\sum_{i\neq j}^{3} y_i a_i\, y_j a_j$

$(y_1 a_1 + y_2 a_2 + y_3 a_3)^2 (y_1 a_1 + y_2 a_2 + y_3 a_3)$

$y_1^2 a_1^2 + y_2^2 a_2^2 + y_3^3 a_3^2 +$

$y_1 a_1\, y_2 a_2 + y_1 a_1\, y_3 a_3 +$

$y_2 a_2\, y_1 a_1 + y_2 a_2\, y_3 a_3 +$

$y_3 a_3\, y_1 a_1 + y_3 a_3\, y_2 a_2$

$y \sim U(-1,1)$
$\mathbb{E}[y] = \frac{1-1}{2} = 0$
$\mathrm{Var}[y] = 1/3$

$$\mathbb{E}[a^2(x,y)] = \int_\Gamma a^2(x,y)\, d\pi_y = \int_\Gamma \left(a_0(x) + \sum y_m \overline{a_m}(x)\right)^2 d\pi_y$$

$$= \int_\Gamma \left[a_0^2(x) + \left(\sum_m y_m \overline{a_m}\right)^2 + 2a_0(x)\sum_m y_m \overline{a_m}\right] d\pi_y$$

$$= a_0^2(x) + \int_\Gamma \left(\sum_m y_m^2 \overline{a_m}^2(x) + 2\sum_{m\neq n} y_m \overline{a_m}\, y_n \overline{a_n}\, d\pi_y + 2a_0 \sum_m \overline{a_m} \underbrace{\int_\Gamma y_m\, d\pi_y}_{=0}\right.$$

$$= a_0^2(x) + \sum_m \overline{a_m}^2 \underbrace{\int_\Gamma y_m^2\, d\pi_y}_{\mathbb{E}[y_m^2]} + 2\sum_{m\neq n} \overline{a_m} a_m \underbrace{\int_\Gamma y_m y_n\, d\pi_y}_{= \mathrm{cov}[y_m y_n] = 0 \text{ as indipend.}}$$

$$= a_0^2(x) + \frac{1}{3}\sum_m \overline{a_m}^2(x) \qquad \equiv \mathrm{Var}(y_m) = 1/3$$

$$\mathbb{E}[a(x,y)] = \int_\Gamma a(x,y)\, d\pi_y = a_0(x) + \sum \overline{a_m}(x) \underbrace{\int_\Gamma y_m\, d\pi_y}_{=0} = c + \varepsilon \equiv a_0(x)$$

$$\mathrm{Var}[a(x,y)] = \mathbb{E}\left[\left(a(\bar{x},y) - \mathbb{E}[a(\bar{x},y)]\right)\left(a(\tilde{x},y) - \mathbb{E}[a(\tilde{x},y)]\right)\right] \leftarrow \bar{x} \equiv \tilde{x}$$

$$= \mathbb{E}[a^2(x,y)] - \underbrace{\mathbb{E}[a(x,y)]^2}$$

$$\equiv \underbrace{\mathbb{E}[a^2(x,y)]} - a_0(x)^2$$

$$\equiv a_0^2(x) + \frac{1}{3}\sum_m \overline{a_m}^2 - a_0(x)^2 = \frac{1}{3}\sum_m \overline{a_m}^2(x)$$

$$\mathrm{Cov}(y_m, y_n) = \mathbb{E}\big[\,(y_m - \overbrace{\mathbb{E}[y_m]}^{=0})(y_n - \overbrace{\mathbb{E}[y_n]}^{=0})\,\big] = \mathbb{E}[y_m y_n] = \int_\Gamma y_m y_n \, d\pi_y$$

$$\mathrm{Var}(y_m) = \mathbb{E}[y_m^2] - \underbrace{\mathbb{E}[y_m]^2}_{=0} = \mathbb{E}[y_m^2]$$

y_m indip. y_n

$\mathbb{E}[y_m]\mathbb{E}[y_n] = 0$ ($=0$, $=0$)

$$\mathrm{Var}(Q(\gamma,y)) = \mathrm{Var}\Big(Q_0(\gamma) + \sum_m y_m e_m\Big) = \mathrm{Var}\Big(\sum_m y_m e_m\Big)$$

$$= \mathbb{E}\Big[\Big(\sum_m y_m e_m - \mathbb{E}\big[\textstyle\sum y_m e_m\big]\Big)^2\Big]$$

$$= \mathbb{E}\Big[\Big(\sum_m y_m e_m - \sum_m e_m \underbrace{\mathbb{E}(y_m)}_{=0}\Big)^2\Big]$$

$$= \mathbb{E}\Big[\Big(\sum_m y_m e_m\Big)^2\Big] = \int_\Gamma \Big(\sum_m y_m e_m\Big)^2 \, d\pi_y$$

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$$\circ\circ\circ = \tfrac{1}{3} \sum_m \overline{a_m(x)}^2$$