Compute the e_{xQ} estimator

$B_0(e_{xQ}, v) = F(v) - B(u_{xp}, v) \quad \forall\, v \in V_{xQ} \qquad Q$ = detail index set (7)

Let $\underbrace{nvtx}_{=:J}$ = #T number of vertices and $\bar{Q} = \#Q$ number of indices in the Q ind.s.

$$e_{xQ} = \sum_{\nu \in Q} \sum_{j=1}^{J} e_{\nu j}\, \varphi_j(x)\, P_\nu(y) \qquad \text{with } \varphi_j \text{ basis function (of } X), \ \nu \in Q$$

$$v = \varphi_i(x)\, P_\mu(y) \qquad \text{with } i \in \{1, \dots, J\}, \ \mu \in Q$$

Then the lhs of (1) is:

$$\begin{aligned} B_0(e_{xQ}, V) &= \sum_{\nu \in Q} \sum_{j=1}^{J} \underbrace{B_0(\varphi_j(x) P_\nu(y),\ \varphi_i(x) P_\mu(y))}\, e_{\nu j} \\ &= \sum_{\nu \in Q} \sum_{j=1}^{J} \left(\int_\Gamma \int_D a_0(x)\, \nabla\varphi_j(x) \cdot \nabla\varphi_i(x)\, dx\, P_\nu(y) P_\mu(y)\, d\pi(y) \right) e_{\nu j} \\ &= \sum_{\nu \in Q} \sum_{j=1}^{J} [k_0]_{ij} \langle P_\nu, P_\mu \rangle_\pi\, e_{\nu j} \qquad , \quad \mu \in Q \end{aligned} \tag{2}$$

where $[K_0]_{ij} := \int_D a_0(x)\, \nabla\varphi_j(x) \cdot \nabla\varphi_i(x)\, dx$, $\langle P_\nu, P_\mu \rangle_\pi := \int_\Gamma P_\nu(y) P_\mu(y)\, d\pi(y) = \delta_{\nu\mu} =: [G_0]_{\mu}$

Since $G_0 = Id$ as $[G_0]_{\nu\mu} = \delta_{\nu\mu}$, (2) results to be a diagonal block system, where each block is the matrix $K_0 \equiv [K_0]_{ij}$:

$$B = \begin{pmatrix} K_0 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & K_0 \end{pmatrix} \qquad \text{with } \operatorname{size}(K_0) = J \times J$$

$\leftarrow \bar{Q} \rightarrow$

and then $\operatorname{size}(B) = (J\bar{Q}) \times (J\bar{Q})$ (3)

Now, the rhs. Check $F(v)$ first:

$$F(v) = \int_\Gamma \int_D f_0(x)\, \varphi_i(x) P_\nu(y)\, d\pi y + \sum_{m=1}^{M} \int_\Gamma \int_D y_m\, f_m(x) \varphi_i(x)\, P_\nu(y)\, d\pi y \ , \quad \begin{matrix} i = 1 \dots J \\ \nu \in Q \end{matrix}$$

Since we assume the source to be deterministic $\Rightarrow f_m = 0 \ \forall\, m \geq 1$, then:

$$F(v) = \int_D f_0(x) \varphi_i(x)\, dx \underbrace{\int_\Gamma P_\nu(y)\, d\pi y} \equiv \int_D f_0(x) \varphi_i(x)\, dx\, \langle P_0, P_\nu \rangle_\pi \quad \forall\, \nu \in Q$$

and since $0 \notin Q$ we have $\langle P_0, P_\nu \rangle_\pi = \delta_{0\nu} \equiv 0 \Rightarrow F(V) = 0$ (4)

Due to (4), $F(V)$ is then a JQ vector of zeros.

Check now $B(U_{xP}, V)$ with $U_{xP} \in V_{xP}$, $V \in V_{xQ}$.

$$\begin{cases} U_{xP} = \sum_{\nu \in P} \sum_{j=1}^{J} U_{\nu j}\, \varphi_j(x)\, P_\nu(y), & \varphi_j \text{ basis functions of } X \\ v = \varphi_i(x)\, P_\mu(y), & v \in V_{xQ}, \quad \mu \in Q \end{cases}$$

$$\Rightarrow B(U_{xP}, V) = \sum_{\nu \in P} \sum_{j=1}^{J} B(\varphi_j(x) P_\nu(y), \varphi_i(x) P_\mu(y))\, U_{\nu j}$$

$$= \sum_{\nu \in P} \sum_{j=1}^{J} \left(\int_\Gamma \int_D a(x,y) \nabla\varphi_j \cdot \nabla\varphi_i \, dx\, P_\nu(y) P_\mu(y) \, d\pi y \right) U_{\nu j}$$

$$= \sum_{\nu \in P} \sum_{j=1}^{J} \left(\int_D a_0(x) \nabla\varphi_j \cdot \nabla\varphi_i \, dx \int_\Gamma P_\nu(y) P_\mu(y) \, d\pi y + \right.$$

$$\left. + \sum_{m=1}^{M} \int_D a_m(x) \nabla\varphi_j \cdot \nabla\varphi_i \, dx \int_\Gamma y_m P_\nu(y) P_\mu(y) \, d\pi y \right) U_{\nu j}, \quad \begin{matrix} i = 1 \dots J \\ \mu \in Q. \end{matrix}$$

Define:

$$K_m \equiv [k_m]_{ij} = \int_D a_m(x) \nabla\varphi_j \nabla\varphi_i \, dx \qquad [G_m]_{\mu\nu} = \langle y_m P_\nu, P_\mu \rangle_\pi = \int_\Gamma y_m P_\nu P_\mu \, d\pi y$$

then (also using K_0 and G_0 from previous page)

$$= \sum_{\nu \in P} \sum_{j=1}^{J} \left([G_0]_{\mu\nu} [K_0]_{ij} + \sum_{m=1}^{M} [G_m]_{\mu\nu} [k_m]_{ij} \right) U_{\nu j}, \quad i = 1 \dots J, \ \mu \in Q. \quad (5)$$

As seen before, $[G_0]_{\mu\nu} = \delta_{\mu\nu} \equiv 0$ since $\mu \in Q$ and $\nu \in P$ here. Then (5) reduces to:

$$= \sum_{\nu \in P} \sum_{j=1}^{J} \left(\sum_{m=1}^{M} [G_m]_{\mu\nu} [k_m]_{ij} \right) U_{\nu j} \qquad (6)$$

Now, let's check $[G_m]_{\mu\nu}$:

$$[G_m]_{\mu\nu} = \langle y_m P_\nu, P_\mu \rangle_\pi = \int_\Gamma y_m P_\nu(y) P_\mu(y) \, d\pi y$$

$$= \langle y_m \prod_{i=1} L_{\nu_i}(y_i), \prod_{i=1} L_{\mu_i}(y_i) \rangle_\pi$$

$$= \langle y_m L_{\nu_m}(y_m), L_{\mu_m}(y_m) \rangle_{\pi_m} \prod_{i \neq m} \underbrace{\langle L_{\nu_i}(y_i), L_{\mu_i}(y_i) \rangle}_{= \delta_{\nu_i \mu_i}} \qquad (7)$$

Now (7) is non-zero only if $\nu_i = \mu_i$ with $i \neq m$. In such case $\delta_{\nu_i \mu_i} = 1$

$$= \langle y_m L_{\nu_m}(y_m), L_{\mu_m}(y_m) \rangle_{\pi_m} \qquad (8)$$

Using the 3-term recurrence with L being Legendre polynomials

$$y_m L_{\nu_m}(y_m) = (a_{\nu_{m+1}})^{-1} L_{\nu_{m+1}}(y_m) + (a_{\nu_m})^{-1} L_{\nu_{m-1}}(y_m) \qquad (9)$$

we can rewrite (8) as:

$$= \langle (a_{\nu_{m+1}})^{-1} L_{\nu_{m+1}}(y_m) + (a_{\nu_m})^{-1} L_{\nu_{m-1}}(y_m), L_{\mu_m}(y_m) \rangle_{\pi_m}$$

$$\equiv (a_{\nu_{m+1}})^{-1} \delta_{(\nu_{m+1})\mu_m} + (a_{\nu_m})^{-1} \delta_{(\nu_{m-1})\mu_m}$$

$$= \begin{cases} (a_{\nu_{m+1}})^{-1} & \text{if } \nu_i = \mu_i \text{ for each } i \neq m \text{ AND } \nu_{m+1} = \mu_m \\ (a_{\nu_m})^{-1} & \text{if } \nu_i = \mu_i \text{ for each } i \neq m \text{ AND } \nu_{m-1} = \mu_m \\ 0 & \text{otherwise.} \end{cases} \qquad (10)$$

Therefore we have a situation like this

$[G_m]_{\mu\nu} \neq 0$ only if $\nu = (\nu_1, \nu_2, \dots, \nu_{m-1}, \nu_m, \nu_{m+1}, \dots)$

$= (\mu_1, \mu_2, \dots, \mu_{m-1}, \mu_m, \mu_{m+1}, \dots) = \mu$

or $\nu = (\nu_1, \nu_2, \dots, \nu_{m-1}, \nu_m, \nu_{m+1}, \dots)$

$= (\mu_1, \mu_2, \dots, \mu_{m-1}, \mu_m, \mu_{m+1}, \dots) = \mu$

Note that since $\mu \in Q$ and $\nu \in P$ and $\dim(Q) \neq \dim(P)$, then G_m are not square matrices. In fact:

$$G_m = \begin{pmatrix} * & \cdots & * \\ | & & | \\ * & \cdots & * \end{pmatrix} \begin{matrix} \uparrow \\ Q \\ \downarrow \end{matrix} \quad \text{they are } \bar{Q} \times \bar{P} \text{ matrix.} \qquad (11)$$

$\leftarrow P \rightarrow$

That is, coming back (6) we have

$$= \sum_{\nu \in P} \sum_{j=1}^{J} \left(\sum_{m=1}^{M} [G_m]_{\mu\nu} [k_m]_{ij} \right) U_{\nu j}$$

$$= \sum_{m=1}^{M} (G_m \otimes k_m) \hat{U} \qquad \text{with } \hat{U} = \{U_{\nu j}\} = \begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} \leftarrow J \times \bar{P} \text{ vector.} \qquad (12)$$

Note that (12) is a $J \times \bar{Q}$ vector. In fact, due to the Kronecker product trick, if we reshape $\hat{U} = \text{reshape}(\hat{U}, J, \bar{P}) \equiv U$, that is a $J \times \bar{P}$ matrix, we rewrite (12) as:

$$= \sum_{m=1}^{M} K_m U G_m^T \qquad (13)$$

and then note that

$$\Rightarrow \underset{\downarrow}{\overset{\uparrow}{J}} \begin{pmatrix} \leftarrow J \rightarrow \\ K_m \end{pmatrix} \underset{\downarrow}{\overset{\uparrow}{J}} \begin{pmatrix} \leftarrow P \rightarrow \\ U \end{pmatrix} \underset{\downarrow}{\overset{\uparrow}{P}} \begin{pmatrix} \leftarrow \bar{Q} \rightarrow \\ G_m^T \end{pmatrix}, \text{ i.e. dim} = J \times \bar{Q} \text{ matrix}$$

and then we reshape it like reshape($\downarrow$, $J\bar{Q}$, 1), then we the rhs vector of dim. $J \times \bar{Q}$. ∎

Compute the e_{YP} estimator

Formulation for the e_{YP} estimator:

$$B_0(e_{YP}, v) = F(v) - B(U_{XP}, v) \qquad \forall\, v \in V_{YP} \qquad (1)$$

We construct the elementwise error residual problem on each element $K \in T$.
Consider:

$$B_0|_K(e_{YP}|_K, v) = F_K(v) - B|_K(U_{XP}, v) \quad \forall\, v \in V_{YP}|_K = Y|_K \otimes P \qquad (2)$$

Let:

$$\begin{cases} U_{XP} \in V_{XP} \text{ on } K: & U_{XP} = \sum_{\nu \in P} \sum_{j=1}^{3} U_{\nu j}\, \varphi_j(x)\, P_\nu(y), \quad \varphi_j \text{ bas. func. of } X, \\ v \in V_{YP} \text{ on } K: & v = \psi_i(x)\, P_\mu(y), \quad \psi_i \text{ basis funct. of } Y \text{ (midpoint)}, \\ & i = 1,2,3. \\ e_{YP} \in V_{YP} \text{ on } K: & e_{YP} = \sum_{\eta \in P} \sum_{k=1}^{3} e_{\eta k}\, \psi_i(x)\, P_\eta(y), \quad \psi_i \text{ again b.f. of } Y. \end{cases}$$

Check the lhs:

$$\begin{aligned} B_0|_K(e_{YP}, v) &= \sum_{\eta \in P} \sum_{k=1}^{3} B_0|_K(\psi_k(x) P_\eta(y), \psi_i(x) P_\mu(y))\, e_{\eta k}, \qquad i = 1,2,3 \\ &= \sum_{\eta \in P} \sum_{k=1}^{3} \left(\int_\Gamma \int_K a_0(x)\, \nabla\psi_k \cdot \nabla\psi_i \, dx\, P_\eta(y) P_\mu(y)\, d\pi_y \right) e_{\eta k} \\ &= \sum_{\eta \in P} \sum_{k=1}^{3} [k_0]_{ik}\, [G_0]_{\mu\eta}\, e_{\eta k}, \quad i = 1,2,3. \qquad (3) \end{aligned}$$

with $[k_0]_{ik} = \int_D a_0 \nabla\psi_k \cdot \nabla\psi_i \, dx$ and $[G_0]_{\mu\eta} = \langle P_\eta, P_\mu \rangle_\pi = \int_\Gamma P_\eta P_\mu \, d\pi_y = \delta_{\eta\mu}$.

Therefore, since $G_0 \equiv Id$, (3) gives rise to block diagonal matrix with each block to be k_0

$$B := \begin{pmatrix} k_0 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & k_0 \end{pmatrix} \begin{matrix} \uparrow \\ 3\bar{P} \\ \downarrow \end{matrix}$$

$$\leftarrow \bar{P} \rightarrow$$

with k_0 being a 3×3 matrix.
Then B is a $(3\bar{P}) \times (3\bar{P})$ matrix, with $\bar{P} = \dim(P)$.

(4)

Check now the rhs. Let the source f to be deterministic, i.e. $f = f_0 + \sum_m y_m f$ with $f_m = 0 \ \forall m \geq 1$. Then:

$$F|_k(v) = \int_\Gamma \int_K f_0(x)\, \psi_i(x)\, P_\mu(y)\, dx\, d\pi y \qquad i = 1,2,3\ ,\ \mu \in P \qquad (5)$$

For the term $B|_k(u_{xp}, v)$ we first integrate by parts:

$$B|_k(u_{xp}, v) = \int_\Gamma \int_K a(x,y)\, \nabla u_{xp}(x,y) \cdot \nabla v(x,y)\, dx\, d\pi y$$

$$= \int_\Gamma \int_K \nabla \cdot \left(a(x,y)\, \nabla u_{xp}(x,y)\right) v\, dx\, d\pi y$$

$$\underbrace{- \int_\Gamma \int_{\partial K \setminus \partial D} \left(a(x,y)\, \nabla u_{xp} \cdot \vec{n}\right) v(s)\, ds\, d\pi y}$$

$$= -\frac{1}{2} \int_\Gamma \int_{\partial K \setminus \partial D} \left[\!\left[a(x,y) \frac{\partial u_{xp}}{\partial \vec{n}} \right]\!\right] v(s)\, ds\, d\pi y \qquad (6)$$

where we add $-1/2$ and the jump since the integration over each edge $E \in \mathcal{E}$ is done twice (neighbours elements). Now since:

$$\nabla \cdot (a(x,y)\, \nabla u_{xp}) = a(x,y) \underbrace{\Delta u_{xp}}_{=0} + \nabla a(x,y) \cdot \nabla u_{xp} \quad \text{since we use } P_1 \text{ approx.}$$

then:

$$= \int_\Gamma \int_K \left[\nabla a(x,y) \cdot \nabla u_{xp}(x,y)\right] v(x,y)\, dx\, d\pi y - \frac{1}{2} \int_\Gamma \int_{\partial K \setminus \partial D} \left[\!\left[a \frac{\partial u_{xp}}{\partial n} \right]\!\right] v(s)\, ds\, d\pi y$$

$$= \sum_{\nu \in P} \sum_{j=1}^{3} \left(\int_\Gamma \int_K \left[\nabla a \cdot \nabla \varphi_j(x)\, P_\nu(y)\right] \psi_i(x)\, P_\mu(y)\, dx\, d\pi y \right) u_{\nu j}$$

$$- \frac{1}{2} \int_\Gamma \int_{\partial K \setminus \partial D} \left[\!\left[a \sum_{\nu \in P} \sum_{j=1}^{3} u_{\nu j}\, \nabla \varphi_j \cdot \vec{n}_K \right]\!\right] P_\nu(y)\ \psi_i(s) P_\mu(y)\, ds\, d\pi y$$

$$= \sum_{\nu \in P} \sum_{j=1}^{3} \left(\int_\Gamma \int_K \nabla a_0(x) \cdot \nabla \varphi_j(x)\, \psi_i(x)\, dx\, P_\nu(y) P_\mu(y)\, d\pi y + \right.$$

$$\left. \sum_{m=1}^{M} \int_\Gamma \int_K \nabla a_m(x) \cdot \nabla \varphi_j(x)\, \psi_i(x)\, dx\ y_m P_\nu(y) P_\mu(y)\, d\pi y \right) u_{\nu j}$$