$$\begin{cases} \Delta U = f & \text{on } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases} \Rightarrow$$

Find $U \in H_0^1$

$\int_\Omega \nabla U \cdot \nabla v = \int_\Omega f v$, $v \in H_0^1$ test and trial space are the same

$\Rightarrow B(U,v) = F(v)$. (1)

$\Rightarrow$ Galerkin approximation is $\int_\Omega \nabla U_n \cdot \nabla v_n = \int_\Omega f v_n$, $v_n \in X_0^1$, where $X_0^1 \subset H_0^1$ is a subspace of H_0^1 (finite dim.)

Find $U_n \in X_0^1$

$\Rightarrow B(U_n, v_n) = F(v_n)$ (2)

Galerkin orthogonality: We have $B(U,v) = F(v)$, $\forall v \in H_0^1$, and since $X_0^1 \subset H_0^1$, it will holds as well for $v_n \in X_0^1 \Rightarrow B(U, v_n) = F(v_n)$, $\forall v_n \in X_0^1 \subset H_0^1$.

On the other hands we know that (2) holds $\Rightarrow B(U_n, v_n) = F(v_n)$, so

$$\Rightarrow B(U, v_n) - B(U_n, v_n) = F(v_n) - F(v_n) = 0 \Rightarrow \boxed{B(U - U_n, v_n) = 0}$$

where $U - U_n \in H_0^1$ (H_0^1), and $v_n \in X_0^1$ (X_0^1).

Galerkin Orthog.

$$\begin{cases} \Delta U = f & \text{on } \Omega \\ U = g & \text{on } \partial\Omega \end{cases}$$

Let's take again the test space be H_0^1 so that $\varphi = 0$ on $\partial\Omega$

$$\Rightarrow \int_\Omega \nabla U \cdot \nabla v - \int_{\partial\Omega} \underbrace{\varphi}_{0} (\nabla U \cdot \vec{n}) = \int_\Omega f v \Rightarrow \int_\Omega \nabla U \cdot \nabla v = \int_\Omega f v, \quad \forall v \in H_0^1.$$

But now $U \neq 0$ on $\partial\Omega$, so indeed U should be seeked in the space $H_g^1 = \{ U \in H^1 \mid U|_{\partial\Omega} = g \}$

So

Find $U \in H_g^1$ | $B(U,v) = F(v)$, $\forall v \in H_0^1$ (1) $\Leftarrow$ Test and Trial space are different

$\Rightarrow$ Galerkin Approximation: Find $U_n \in X_g^1$ | $B(U_n, v_n) = F(v_n)$, $\forall v_n \in X_0^1$ (2)

From (1), since $X_0^1 \subset H_0^1 \Rightarrow$ we can put $v = v_n \in X_0^1 \subset H_0^1 \Rightarrow B(U, v_n) = F(v_n)$, $\forall v_n \in X_0^1$, $U \in H_g^1$

So:

$B(U, v_n) - B(U_n, v_n) = F(v_n) - F(v_n) = 0 \Rightarrow B(\underbrace{U - U_n}_{\in H_g^1 - U_n|_{\partial\Omega}}, v_n) = 0$

(H_g^1, X_g^1, X_0^1)

$$\|U\|_e^2 = \|U - U_n + U_n\|_e^2 = \|U - U_n\|_e^2 + \|U_n\|_e^2 + 2B(U - U_n, U_n) \neq 0$$

$$\Downarrow$$

$$\boxed{\|U - U_n\|_e^2 = \|U\|_e^2 - \|U_n\|_e^2 - 2B(U - U_n, U_n)}$$

$$= \|U\|_e^2 + \|U_n\|_e^2 - 2B(U, U_n)$$

So here we have $U - U_n \in H_{g-U_n|_{\partial\Omega}}^1$ but $U_n \in X_g^1$! to be 0, it should be in X_0^1.

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$u \sim r^{2/3}$

$u'_r \sim r^{-1/3}$

$\frac{\partial^2 u}{\partial r^2} \sim r^{-4/3}$

$$\int_0^1 x^\alpha dx < +\infty \qquad \alpha > -1$$

$$= \frac{x^{\alpha+1}}{\alpha+1}\Big|_0^1$$

$$\int_D \left|\frac{\partial^2 u}{\partial r^2}\right|^2 r\,dr\,d\theta = \int_0^1 r^{-\frac{8}{3}+1} dr = +\infty \qquad -\frac{8}{3}+1 < -1 \Rightarrow u \notin H^2$$

$$-\frac{8}{3} < -2$$

$$\int_D |\nabla u|^2 dx\,dx \sim \int_0^1 r^{-2/3} \cdot r\,dr = \int_0^1 r^{1/3} dr < +\infty$$

$u \in H^1$ but $u \notin H^2$

$\frac{1}{2} \quad \frac{1}{3}$

$\frac{1}{2.5} = \frac{2}{5}$

$\varphi = \frac{3\pi}{2}$

$\alpha = \frac{\pi}{\varphi} = \frac{\pi \cdot 2}{3\pi} = \frac{2}{3}$

$$u \sim r^{2/3} \xrightarrow{H^s} \partial^s u \sim r^{2/3-s}$$

$$|u|^2_{H^s(D)} \sim \int_0^1 (r^{2/3-s})^2 r\,dr < +\infty$$

$$\frac{4}{3} - 2s + 1 > -1$$

$$2s < 2 + \frac{4}{3} = \frac{10}{3}$$

$$s < \frac{5}{3}$$

$h = N^{-1/2}$

$N^{-\frac{1}{3}}$

$\frac{4}{\sigma^2 6}$ $\quad N^{-\frac{1}{2}}$

$$\|u - u_h\|_{H_0^1(D)} \lesssim h^{5/3-\varepsilon-1} \simeq h^{\frac{2}{3}-\varepsilon} \simeq \left(N^{-1/2}\right)^{\frac{2}{3}-\varepsilon}$$

$$\simeq N^{-\frac{1}{3}+\varepsilon}$$

$0 < s < 1$

$$\|\varphi\|^2 = \iint_{DD} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dx\,dy < \infty$$

Sobolev - Slobodetskij semi-norm

$$H^s := \{ u;\ \|u\|^2_{H^s} = \|u\|^2_{L^2} + |u|^2_{H^s} < +\infty \}$$

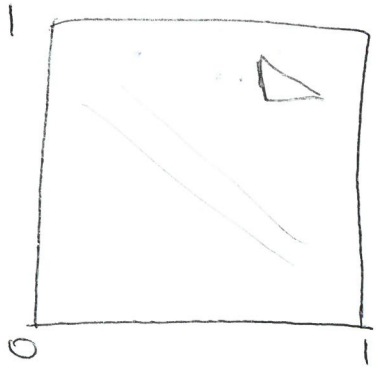
H^0, H^1, H^2 $\qquad H^t = (H^0, H^1)_{\theta,2}$

Interpolation spaces

$N^{-1/2}$ $C \cdot N^{-1/2}$

$a(x,y) = 1 + fx \cdot * fy$

$f(x) = (2 - x^2 - y^2)/8$



Compute FE solutions and error estimates with P1 and P2 for grid parameters 2 to 8.

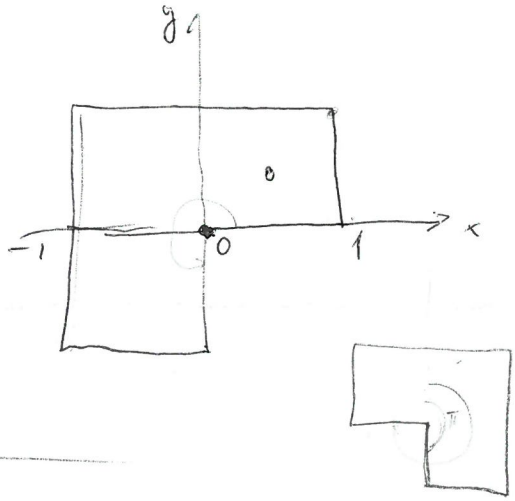
Plot error(s) vs # d.o.f.

2. Solution $u(r,\theta) = r^{2/3} \sin\left(\frac{2\theta+\pi}{3}\right)$.

$r = \sqrt{x^2+y^2}$

θ - angle with the vertical axis

~~$div(a \nabla u) = f$~~



* L-shaped
* $f = 1$
* $a(x,y) = 1 + fx \cdot * fy$ $\Big\}$ $1 - \frac{1}{4}(x^2+y^2)$

$\nabla a = \left(0 - \frac{1}{4}(2x), -\frac{1}{4}(2y)\right)$

$= (-x/2, -y/2)$

$u(r,\theta) = r^{2/3} \sin\left(\frac{2\theta+\pi}{3}\right)$, $a(x,y) \to a(r) = 1 - \frac{y}{4} r^2 \cdot \left[1 - \frac{1}{4}(x^2+y^2)\right]$

$-div(a(r) \nabla u(r,\theta)) = \frac{1}{3} r^{2/3} \sin\left(\frac{2\theta+\pi}{3}\right) = \frac{1}{3} u(r,\theta) = f$

$\|u - u_h\|_{H_0^1(D)} \leq C h^{s-1} \|u\|_{H^s(D)}$ $\quad 1 < s \leq 2$

$\frac{2}{3} \cdot \frac{1}{4}$