• We want to consider ALG 2, the one using the "marked versions" of the estimators.

1. Consider the computable η estimator, that is (3)

$$\eta := \left(\| e_{yp} \|^2_{B_0} + \| e_{x\mathcal{Q}} \|^2_{B_0} \right)^{1/2} \qquad (1)$$

where, here, $\| e_{x\mathcal{Q}} \|^2_{B_0} := \sum_{\mu \in \mathcal{Q}} \| e^{(\mu)}_{x\mathcal{Q}} \|^2_{B_0}$ where $\mathcal{Q}$ is the finite index set as defined by (5.1) in the paper (i.e. the computable finite $\mathcal{Q}$ index-set)

2. The estimates of the error reduction apply straightforward to the "computable" setting. The original estimates consider $\| e_{x\mathcal{Q}} \|_{B_0}$, where $\mathcal{Q}$ is in principle infinite, e.g. $\mathcal{Q} = \mathcal{Q}_\infty$, and the $P^* \equiv P^*_\infty := P \cup \mathcal{Q}_\infty$. So it holds

$$\lambda \| e_{x\mathcal{Q}_\infty} \|^2_{B_0} \leq \| U_{xP^*_\infty} - U_{xP} \|^2_B \qquad (2)$$

Consider how the finite computable $\mathcal{Q}$ index set as explained in point 1. Then we consider $P^* = P \cup \mathcal{Q}$ and it holds

$$\lambda \| e_{x\mathcal{Q}} \|^2_{B_0} \leq \| U_{xP^*} - U_{xP} \|^2_B \qquad (3)$$

3. Dörfler Marking: given $\vartheta_\mu \in (0,1)$ let $M_\mathcal{Q}$ be the minimal set of indices of $\mathcal{Q}$ (the computable one as explained in 1.) such that

$$\| e^{M_\mathcal{Q}}_{x\mathcal{Q}} \|^2_{B_0} := \sum_{\mu \in M_\mathcal{Q}} \| e^{(\mu)}_{x\mathcal{Q}} \|^2_{B_0} \geq \vartheta_\mu \sum_{\mu \in \mathcal{Q}} \| e^{(\mu)}_{x\mathcal{Q}} \|^2_{B_0} = \vartheta_\mu \| e_{x\mathcal{Q}} \|^2_{B_0} \qquad (4)$$

Since $M_\mathcal{Q}$ is a (finite) subset of the (finite) $\mathcal{Q}$, then $\| e^{M_\mathcal{Q}}_{x\mathcal{Q}} \|^2_{B_0} \leq \| e_{x\mathcal{Q}} \|^2_{B_0}$. Moreover, the error reduction in this case reads as

$$\lambda \| e^{M_\mathcal{Q}}_{x\mathcal{Q}} \|^2_{B_0} \leq \| U_{xP^*} - U_{xP} \|^2_B \qquad (5)$$

where the enriched index set P^* is given by P plus the marked indices of $\mathcal{Q}$, i.e. $P^* := P \cup M_\mathcal{Q}$, with $M_\mathcal{Q}$ defined by (4).
↳ (i.e. U_{xP^*} is the enriched solut. of our AFEM ALG2)

Dörfler Marking: given $\vartheta_x \in (0,1)$ let M be the minimal set of elements $k \in \mathcal{T}$ such that

$$\| e^{M}_{yp} \|^2_{B_0} := \sum_{k \in M} \| e_{yp}|_k \|^2_{B_0} \geq \vartheta_x \sum_{k \in \mathcal{T}} \| e_{yp}|_k \|^2_{B_0} = \vartheta_x \| e_{yp} \|^2_{B_0} \qquad (6)$$

4. Consider the Error bound for the true error $u - u_{xp}$:

$$\|u - u_{xp}\|_B^2 \leq \frac{\Lambda}{(1-\beta^2)(1-\gamma^2)} \underbrace{\left(\|e_{yp}\|_{B_0}^2 + \|e_{xq}\|_{B_0}^2 \right)}_{=:\eta^2} \quad (7)$$

where η^2 is the computable estimator (for the finite index set). β

(*) See pag 7 for a remark for β

Multiply (7) by θ_μ and use (4):

$$\theta_\mu \|u - u_{xp}\|_B^2 \leq \frac{\Lambda}{(1-\beta^2)(1-\gamma^2)} \left(\theta_\mu \|e_{yp}\|_{B_0}^2 + \|e_{xq}^{M_q}\|_{B_0}^2 \right) \quad (8)$$

Multiply (8) by λ and use error reduction (5):

$$\lambda\theta_\mu \|u - u_{xp}\|_B^2 \leq \frac{\Lambda}{(1-\beta^2)(1-\gamma^2)} \left(\theta_\mu \lambda \|e_{yp}\|_{B_0}^2 + \|u_{xp^*} - u_{xp}\|_B^2 \right)$$

from which we get:

$$-\|u_{xp^*} - u_{xp}\|_B^2 \leq -\lambda\theta_\mu \frac{(1-\beta^2)(1-\gamma^2)}{\Lambda} \|u - u_{xp}\|_B^2 + \lambda\theta_\mu \|e_{yp}\|_{B_0}^2 \quad (9)$$

5. Galerkin orthogonality: $u_{xp^*} \rightsquigarrow$ depends on $p^* = p \cup M_q$, that is is the enriched solution arising from our AFEM ALG2

$$\|u - u_{xp}\|_B^2 = \|u - u_{xp^*}\|_B^2 + \|u_{xp^*} - u_{xp}\|_B^2$$

$$\|u - u_{xp^*}\|_B^2 = \|u - u_{xp}\|_B^2 - \|u_{xp^*} - u_{xp}\|_B$$

Add and subtract $\pm \alpha \|u_{xp^*} - u_{xp}\|_B^2$, for some $\alpha \in (0,1)$ to be chosen:

$$\|u - u_{xp^*}\|_B^2 = \|u - u_{xp}\|_B^2 \underbrace{- \alpha \|u_{xp^*} - u_{xp}\|_B}_{} - (1-\alpha)\|u_{xp^*} - u_{xp}\|_B^2 \quad (10)$$

Now, multiply (9) by α and bound (10):

$$\|u - u_{xp^*}\|_B^2 \leq \|u - u_{xp}\|_B^2 - \alpha\lambda\theta_\mu \frac{(1-\beta^2)(1-\gamma^2)}{\Lambda} \|u - u_{xp}\|_B^2 + \alpha\lambda\theta_\mu \|e_{yp}\|_{B_0}^2 - (1-\alpha)\|u_{xp^*} - u_{xp}\|_B^2$$

$$\|u - u_{xp^*}\|_B^2 \leq \underbrace{\left[1 - \alpha\lambda\theta_\mu \frac{(1-\beta^2)(1-\gamma^2)}{\Lambda} \right]}_{=:c_1} \|u - u_{xp}\|_B^2 + \alpha\lambda\theta_\mu \|e_{yp}\|_{B_0}^2 - (1-\alpha)\|u_{xp^*} - u_{xp}\|_B^2 \quad (11)$$

6. Now there is the choice of the algorithm. During the polynomial enrichment, it happens that $\| e_{x\alpha}^{M\alpha} \|_{B_0}^2 \geq \| e_{yp}^{M} \|_{B_0}^2$, and using (6) we get (5)

$$\underbrace{\| e_{x\alpha}^{M\alpha} \|_{B_0}^2 \geq \| e_{xp}^{M} \|_{B_0}^2}_{\text{ALG 2}} \geq \underbrace{\theta_x \| e_{yp} \|_{B_0}^2}_{\text{Dörfler (6)}} \qquad (12)$$

Therefore, by using (5) we deduce that

$$\lambda \theta_x \| e_{yp} \|_{B_0}^2 \leq \| U_{xp^*} - U_{xp} \|_B^2 \qquad (13)$$

Multiply (13) by $(1-\alpha)$, we get

$$-(1-\alpha) \| U_{xp^*} - U_{xp} \|_B^2 \leq -(1-\alpha) \lambda \theta_x \| e_{yp} \|_{B_0}^2 \qquad (14)$$

7. We can use (14) to bound the last term in (11):

$$\| U - U_{xp^*} \|_{B_0}^2 \leq c_1 \| U - U_{xp} \|_B^2 + \alpha \lambda \theta_\mu \| e_{yp} \|_{B_0}^2 - (1-\alpha) \lambda \theta_x \| e_{yp} \|_{B_0}^2 \qquad (15)$$

By choosing now the value α such that:

$$\alpha \lambda \theta_\mu = (1-\alpha) \lambda \theta_x \quad \Rightarrow \quad \boxed{\alpha := \frac{\theta_x}{\theta_x + \theta_\mu}} \qquad (16)$$

then the 2nd and 3rd term in (15) will cancel and we get:

$$\boxed{\| U - U_{xp^*} \|_{B_0}^2 \leq c_1 \| U - U_{xp} \|_B^2} \qquad (17)$$

Error Reduction for Polynomial enrich.

where c_1 is defined by

$$c_1 := 1 - \alpha \lambda \theta_\mu \frac{(1-\beta^2)(1-\gamma^2)}{\Lambda} \equiv 1 - \underbrace{\frac{\theta_x \theta_\mu}{\theta_x + \theta_\mu} \frac{\lambda}{\Lambda} (1-\beta^2)(1-\gamma^2)}_{=: c_2} \qquad (18)$$

with $c_2 < 1$, which implies $c_1 < 1$. □

END ALG 2

- If we want to use now ALG 1, all the steps up to 5. remain the same, since we still consider the computable estimator η, that is with $\mathcal{Q}$ being a finite index set, and the polynom. enrichment still involve only adding the marked indices in $M_{\mathcal{Q}}$, i.e. $P^* = P \cup M_{\mathcal{Q}}$ (see 3.).

Therefore, since condition of ALG 1 reads as $\|e_{x\mathcal{Q}}\|^2_{B_0} \geq \|e_{yP}\|^2_{B_0}$, that is we are not comparing any more the "marked versions" of the estimators, we can do:

ALG 1 condition: $\|e_{x\mathcal{Q}}\|^2_{B_0} \geq \|e_{yP}\|^2_{B_0} \Rightarrow \vartheta_\mu \|e_{x\mathcal{Q}}\|^2_{B_0} \geq \vartheta_\mu \|e_{yP}\|^2_{B_0}$

and now use Dörfler (4):

$$\|e^{M_{\mathcal{Q}}}_{x\mathcal{Q}}\|^2_{B_0} \geq \vartheta_\mu \|e_{x\mathcal{Q}}\|^2_{B_0} \geq \vartheta_\mu \|e_{yP}\|^2_{B_0}. \qquad (19)$$

Hence, by using the error reduction property (5) we get the bound:

$$\lambda \vartheta_\mu \|e_{yP}\|^2_{B_0} \leq \|U_{xP^*} - U_{xP}\|^2_B$$

Then, multiply $-(1-\alpha)$ both sides:

$$\Rightarrow \quad -(1-\alpha)\|U_{xP^*} - U_{xP}\|^2_B \leq -(1-\alpha)\lambda\vartheta_\mu \|e_{yP}\|^2_{B_0} \qquad (20)$$

8. Now we can continue as in previous section 7. and use (20):

$$\|U - U_{xP^*}\|^2_B \leq G \|U - U_{xP}\|^2_B + \alpha\lambda\vartheta_\mu \|e_{yP}\|^2_{B_0} - (1-\alpha)\lambda\vartheta_\mu \|e_{yP}\|^2_{B_0} \qquad (21)$$

and now by choosing $\alpha := 1/2$ the 2nd and 3rd term in (21) cancel and we obtain:

$$\|U - U_{xP^*}\|^2_B \leq c_1 \|U - U_{xP}\|^2_B \qquad (22)$$

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with $c_1 := 1 - \alpha\lambda\vartheta_\mu \frac{(1-\beta^2)(1-\gamma^2)}{\Lambda} \equiv 1 - \underbrace{\frac{1}{2}\vartheta_\mu \frac{\lambda}{\Lambda}(1-\beta^2)(1-\gamma^2)}_{=: c_2}$ (23)

again, with $c_2 < 1$ which then implies $c_1 < 1$. □

END ALG 1

NOTE on ALG2 proof: we proved ALG2 by using the algorithm decision condition $\| e_{xQ}^{M_Q} \|_{B_0}^2 \geq \| e_{yP}^{M} \|_{B_0}^2$ where $M \subset T$ is the set of marked elements defined by (6). Indeed, ALG2 considers the overall set of marked elements, i.e. the "refined" elements, which is the set $R \supseteq M$ bigger than M. However, it holds

$$\| e_{yP}^{R} \|_{B_0}^2 := \sum_{K \in R} \| e_{yP}|_K \|_{B_0}^2 \geq \sum_{K \in M} \| e_{yP}|_K \|_{B_0}^2 =: \| e_{yP}^{M} \|_{B_0}^2 . \quad (24)$$

Therefore, the whole proof still follows with the right decision-choice

$$\boxed{\| e_{xQ}^{M_Q} \|_{B_0}^2 \geq \| e_{yP}^{R} \|_{B_0}^2} \quad (25)$$

as the corresponding relation (12) will become:

$$\underbrace{\| e_{xQ}^{M_Q} \|_{B_0}^2 \geq \| e_{yP}^{R} \|_{B_0}^2}_{\text{ALG2}} \geq \underbrace{\| e_{yP}^{M} \|_{B_0}^2 \geq \theta_X \| e_{yP} \|_{B_0}^2}_{\text{Dörfler (6)}} \quad (26)$$

(24)

□

NOTE on β appearing first in (7) for the error-upper bound: we should consider the $\beta \in (0,1)$ constant as the constant appearing in the deterministic saturation assumption, i.e. $\beta = \beta_h \in (0,1)$ (See inequality (2) of the other set of notes)

ISSUE with algorithm 3, i.e. the one using $\mathcal{T}_{ref}$ and providing the exact error reduction $\sqrt{\lambda} \| e_{yref P} \|_{B_0} \leq \| U_{x^+_P} - U_{xP} \|_B \leq \frac{\sqrt{\Lambda}}{\sqrt{1-\delta^2}} \| e_{yref P} \|_{B_0}$.

⇒ If I try to apply the same argument / technique for ALG3 I got some trouble when obtaining eq. (9) (well, the corresponding one)

↳

Starting again from (7) I would have:

$$\| U - U_{xp} \|_B^2 \leq \frac{\Lambda}{(1-\beta^2)(1-\gamma^2)} \left(\| e_{yp} \|_{B_0}^2 + \| e_{xa} \|_{B_0}^2 \right)$$

$$\Rightarrow \quad \vartheta_x \| U - U_{xp} \|_B^2 \leq \frac{\Lambda}{(1-\beta^2)(1-\gamma^2)} \left(\vartheta_x \| e_{yp} \|_{B_0}^2 + \vartheta_x \| e_{xa} \|_{B_0}^2 \right)$$

$$\Rightarrow \text{(Dörfler (6))} \quad \vartheta_x \| U - U_{xp} \|_B^2 \leq \frac{\Lambda}{(1-\beta^2)(1-\gamma^2)} \left(\| e_{yp}^M \|_{B_0}^2 + \vartheta_x \| e_{xa} \|_{B_0}^2 \right) \quad (27)$$

and now I cannot use the corresponding error-reduction estimate since $\| e_{yp}^M \|_{B_0}$ doesn't reflect exactly the error reduction achieved if the mesh refinement (based on Y_{ref} space) is actually done ...

... we should kind of prove any relation between $\| e_{yp}^M \|_{B_0}$ and $\| e_{y_{ref}p} \|_{B_0}$, that at first sight could be:

$$\| e_{y_{ref}p} \|_{B_0}^2 \leq \| e_{yp}^M \|_{B_0}^2 \leq \| e_{yp}^\ell \|_{B_0}^2 \quad (28)$$

this is MAYBE true (at least confirmed by experiments) and also intuitively makes sense since for a given marked element $k \in \mathcal{T}$, $\| e_{yp|k} \|_{B_0}$ is based on the 3 bubble midpoints functions, while $\| e_{y_{ref}p}^{(k)} \|_{B_0}$ depends on the actual marked edges (≤ 3 anyway!)

this is surely true:
$\sum_{k \in M} \| e_{yp|k} \|_{B_0}^2 \leq \sum_{k \in \mathcal{T}} \| e_{yp|k} \|_{B_0}^2$

However, maybe (28) is not still enough to prove anything ...
don't know ...