w

$$\begin{cases} -\nabla\cdot(a(x,y)\nabla u(x,y)) = f(x) & x\in D,\ y\in\Gamma \\ u(x,y) = g(x) & x\in\partial D,\ y\in\Gamma \end{cases}$$

∂D = entire Dirichlet boundary

Let $g\in H^1(\partial D)$

1) Weak solution $u(x,y)\in H^1_g(D)\otimes L^2_\pi(\Gamma)$ s.t.

$$B(u,v) = F(v) \quad \forall\ v\in H^1_0(D)\otimes L^2_\pi(\Gamma) \qquad (1)$$

Note that $H^1_g(D) = \{ w\in H^1(D) \text{ s.t. } w = g \text{ on } \partial D\}$.

Denote

$$H^1_g(D)\otimes L^2_\pi(\Gamma) =: V^g \qquad (2)$$

$$H^1_0(D)\otimes L^2_\pi(\Gamma) =: V^0 \qquad (3)$$

Inhomogeneous Dirichlet Data case

Suppose now that, since we have $g\in H^1(\partial D)$ (i.e. on the 1D manifold it is continuous), we have g_X to be the nodal interpolant of g on ∂D, which is then well defined. We have to approximate $g\approx g_X$.

2) Weak Galerkin solution: $u_{XP}\in X_g\otimes P_p$ s.t.

$$B(u_{XP}, v) = F(v) \quad \forall\ v\in X_0\otimes P_p \qquad (4)$$

(*)

Note that since $X_g = \{ w\in C^0(\bar D) \text{ s.t. } w|_{\partial D} = g_X \text{ and } w|_k\in P_1(k)\}$ it means that our solution $u_{XP} = g_X$ on ∂D. In particular $u_{XP}\equiv g$ over the nodal boundary points. Notice that while $X_0\subset H^1_0(D)$, we have that $X_g\not\subset H^1_g(D)$ since $w\in X_g$ satisfies the boundary condition approximately (only at boundary nodal points). (*)

NOTE: Notation. Since u_{XP} has trace equal to g_X, we should call the space X_g as X_{g_X} indeed!!

Given the mesh $\mathcal{T}$ of the domain D we have:

$$X_0 := \overbrace{\text{span}}^{\text{span}}\{\varphi_1,\dots,\varphi_N\} \qquad (5)$$

$$X_g := \text{span}\{\varphi_1,\dots,\varphi_N,\varphi_{N+1},\dots\varphi_{N+N_\partial}\} \qquad (6)$$

with N = # internal nodes, N_∂ = # boundary nodes, φ_j Lagrange basis function (P1) associated with the j-th node of the mesh $\mathcal{T}$. Then the nodal interpolant g_X is nothing but

$$g_X(x) = \sum_{i=N+1}^{N+N_\partial} g_i\,\varphi_i(x) \quad \forall\ x\in\partial D \ (\text{and } g_i = g(N_i)) \qquad (7)$$

(↑ note this! which means $\Rightarrow \varphi_i|_{\partial D}(x)$)

Denote:

$$\begin{cases} X_g \otimes P_p =: V^g_{xp} & (8) \\ X_0 \otimes P_p =: V^0_{xp} & (9) \end{cases}$$

Note again that while $V^0_{xp} \subset V^0$, $V^g_{xp} \not\subset V^g$ since $X_g \not\subset H^1_g(\Omega)$.

Choosing in (1) a test function $v \in V^0_{xp} \subset V^0$, and subtracting on both sides $B(U_{xp}, v)$ we obtain:

$$\begin{aligned} B(U,v) - B(U_{xp}, V) &= F(V) - \underbrace{B(U_{xp}, V)}_{} \qquad \forall\, v \in V^0_{xp} \\ &= F(V) - F(V) = 0 \qquad \forall\, v \in V^0_{xp} \end{aligned}$$

which gives

$$B(e, v) = 0 \quad (\text{with } e := U - U_{xp}) \quad \forall\, v \in V^0_{xp} \qquad (10)$$

which is the Galerkin orthogonality. The "problem", respect to the pure homogeneous case, is that e doesn't belong to the space V^0_{xp} here. In fact, $U \in V^g$ and $U_{xp} \in V^g_{xp}$ which have respectively trace equal to g and g_x. Then $e = U - U_{xp} \in V^{g-g_x} =: H^1_{g-g_x}(\Omega) \otimes L^2_\pi(\Gamma)$.

Note that $g - g_x \in H^{\frac{1}{2}}(\partial\Omega)$

But is this really correct?? If $V^g_{xp} \not\subset V^g$ how can the difference $U - U_{xp} \in V^{g-g_x}$? Or maybe yes, in fact it doesn't stay in V^g but in V^{g-g_x} ... confuse ...

Recall:

(11) $B(U - U_{xp}, v) = 0 \quad \forall\, v \in V^0_{xp}$ (but note that $U_{xp} \in V^g_{xp}$, $U - U_{xp} \in V^{g-g_x}$)

We see that the true error $e = U - U_{xp} \in V^{g-g_x}$ satisfies the following:

$$B(e, V) = F(V) - B(U_{xp}, V) \quad \forall\, v \in V^0 \qquad (12)$$

(just substitute $U = e + U_{xp}$ in (1))

Since we cannot discretize (approximate) the true error e using the same mesh $\mathcal{T}$ and P_p (i.e. using the space $V^0_{\mathcal{T}p}$), as we will get the trivial error $e=0$ (see Galerkin orthogonality (11)), we have to enrich the discrete space. Let:

$$\begin{cases} X^*_g := X_g \oplus Y_g & (Y_g \cap X_g = \{0\}) \quad (13) \\ X^*_0 := X_0 \oplus Y_0 & (Y_0 \cap X_0 = \{0\} \text{ and also now } Y_0 \subset H^1_0(\Omega)) \quad (14) \end{cases}$$

$$\left\{ P^*_p := P_p \oplus P_q \quad (P^* = P \cup Q,\ Q \subset \mathcal{T},\ P \cap Q = \emptyset,\ P_p \cap P_q = \{0\}) \right. \quad (15)$$

Then consider the spaces $\boxed{X_g \otimes P_Q =: V^g_{\mathcal{T}Q}}$, $\boxed{Y_g \otimes P_P =: V^g_{YP}}$,

$\boxed{X_0 \otimes P_Q =: V^0_{\mathcal{T}Q}}$, $\boxed{Y_0 \otimes P_P =: V^0_{YP}}$

and define:

$$\begin{cases} V^{g*}_{\mathcal{T}P} := V^g_{\mathcal{T}P} \oplus V^g_{YP} \oplus V^g_{\mathcal{T}Q} \text{ or } 0? & (16) \\ V^{0*}_{\mathcal{T}P} := V^0_{\mathcal{T}P} \oplus V^0_{YP} \oplus V^0_{\mathcal{T}Q} & (17) \end{cases}$$

maybe this has to be 0 indeed because already $V^g_{\mathcal{T}P} \oplus V^g_{YP}$ incorporate the g^*_x Dirichlet data: but then what space is it? $V^0_{\mathcal{T}Q} = X_0 \otimes P_Q$, as here?

Also in the code: what boundary conditions to apply in X_Q estimation here?

Then we have:

Find $u^*_{\mathcal{T}P} \in V^{g*}_{\mathcal{T}P}$ s.t. $B(u^*_{\mathcal{T}P}, v) = F(v)$ $\forall v \in V^{0*}_{\mathcal{T}P}$ (18)

Note that the enriched solution $u^*_{\mathcal{T}P}$ has now trace $u^*_{\mathcal{T}P}|_{\partial\Omega} = g^*_x$ that is the nodal interpolant of g on the boundary $\partial\Omega$ given the "enriched" mesh $\mathcal{T}^*$ associated with the previous one plus all midpoints. Of course, g^*_x is still well-defined.

Consider now the "error" $e^* := u^*_{\mathcal{T}P} - u_{\mathcal{T}P}$. Note that $u_{\mathcal{T}P} \in V^g_{\mathcal{T}P}$ and its trace is g_x, $u^*_{\mathcal{T}P} \in V^{g*}_{\mathcal{T}P}$ and its trace is g^*_x. Also $V^g_{\mathcal{T}P} \not\subset V^{g*}_{\mathcal{T}P}$. FALSE, the sign is $\not\subset$

Then $e^* \in V^{g^*_x - g_x}_{\mathcal{T}P} \subset C\, H^1_{g^*_x - g_x}(\Omega) \otimes L^2_\Pi(\Gamma) =: V^{g^*_x - g_x}$. (19)

Using now (18) and writing $u^*_{xp} = e^* + u_{xp}$ we see that e^* solves the problem

$$b(e^*, v) = F(v) - b(u_{xp}, v) \quad \forall\, v \in V^{0*}_{xp} \qquad (20)$$

I still don't know if it is necessary and has to be used at some point, but just let me remark this. If we let $e^* \in V^{g^*_x - g_x}_{xp}$ to be written as $e^* = e^{0*} + e^{*g}$, with $e^{0*} \in V^{0*}_{xp}$ and $e^{*g} \in V^{g^*_x - g_x}_{xp}$ then (20) becomes

$$b(e^{0*}, v) = F(v) - b(u_{xp}, v) - b(e^{*g}, v) \quad \forall\, v \in V^{0*}_{xp} \qquad (21)$$

which is now a "symmetric" formulation.

1° Error Bound Although e^* is not going to being computed, we should show that is related to e somehow, basically $\|e\|_B \simeq \|e^*\|_B$. To tackle this let me recall some G.O. properties:

$$\begin{cases} b(u - u_{xp}, v) = 0 \quad \forall\, v \in V^0_{xp} & (22) \\ b(u - u^*_{xp}, v) = 0 \quad \forall\, v \in V^{0*}_{xp} & (23) \\ b(u^*_{xp} - u_{xp}, v) = 0 \quad \forall\, v \in V^0_{xp} & (24) \end{cases}$$

Now (I'm going to have other problems with notation) let U^*_{xp} (it's capital U) be the unique Galerkin solution $\in V^{*g}_{xp}$ BUT having the trace equal to g_x (rather than g^*_x). Note that this is clearly obvious in two cases: if the polynomial enrichment is done then the trace of U^*_{xp} is still the "previous" one g_x, and if the optical refinements are done only for internal edges, again the trace of the enriched solution would not change.

If instead some edge of the boudary is bisected, we basically solve the Galerkin formulation imposing a "wrong" trace equal to g_X (i.e. instead of having $U^*_{xp}(x_i) = g(x_i)$ we let $U^*_{xp}(x_i) = g_X(x_i)$ where x_i is a boudary new node (a midpoint))

However, we will have $U^*_{xp} - u_{xp} \in V^0_{xp}$ (25)

Consider the following:

$$2\,|B(u - u^*_{xp},\, u^*_{xp} - u_{xp})| = 2\,|B(u - u^*_{xp},\, u^*_{xp} - U^*_{xp} + U^*_{xp} - u_{xp})|$$

$$= 2\,|B(u - u^*_{xp},\, u^*_{xp} - U^*_{xp}) + \underbrace{B(u - u^*_{xp},\, U^*_{xp} - u_{xp})}_{\overset{(23)}{=} 0}|$$

$$= 2\,|B(u - u^*_{xp},\, u^*_{xp} - U^*_{xp})|$$

$$\leq \alpha \,\|u - u^*_{xp}\|^2_B + \alpha^{-1}\,\|u^*_{xp} - U^*_{xp}\|^2_B \quad \text{for } \alpha > 0 \qquad (26)$$

where we used Young inequality (stupidly: $2ab \leq \varepsilon^{-1}a^2 + \varepsilon b^2$, $\varepsilon > 0$).

Consider now a "lift" operator $\mathcal{L}: V^{\tilde{g}}_{xp}|_{\partial D} \to V^{\tilde{g}}_{xp}$ simply defined by

$$\mathcal{L}(w_{xp}|_{\partial D}) = w_{xp} \quad \forall\, w_{xp} \in V^{\tilde{g}}_{xp} \qquad (27)$$

whatever $\tilde{g}$ (the trace) is. Indeed I believe we can consider a fully non-parametric operator, that is, $\mathcal{L}: X_{\tilde{g}}|_{\partial D} \to X_{\tilde{g}}$ (28).

The point is now: if I consider the following function:

$$\vartheta^*_{xp} := \underbrace{(u^*_{xp} - U^*_{xp})}_{\text{has trace } g^*_x - g_x} - \underbrace{\mathcal{L}(g^*_x - g_x)}_{\text{has trace } g^*_x - g_x} \in V^{*0}_{xp} \qquad (29)$$

(note that in fact, since $g^*_x - g_x$ is non parametric I could directly use (28), maybe? Should does not matter...)

then I can do:

$$0 = 0 - 0 \overset{(23)}{\equiv} B(u - U^*_{xp}, v) - B(u - U^*_{xp}, v)$$

$$= B(U^*_{xp} - U^*_{xp}, v) \quad \forall\, v \in V^{0*}_{xp} \,. \qquad (30)$$

Choose v in (30) to be v^*_{xp} in (29):

$$0 = B(U^*_{xp} - U^*_{xp},\ U^*_{xp} - U^*_{xp} - \ell(g^*_x - g_x))$$

$$= \| U^*_{xp} - U^*_{xp} \|^2_B - B(U^*_{xp} - U^*_{xp},\ \ell(g^*_x - g_x)) \Rightarrow$$

$$\Rightarrow \| U^*_{xp} - U^*_{xp} \|^2_B \leq \| U^*_{xp} - U^*_{xp} \|_B \| \ell(g^*_x - g_x) \|_B \Rightarrow$$

$$\Rightarrow \| U^*_{xp} - U^*_{xp} \|_B \leq \| \ell(g^*_x - g_x) \|_B$$

$$= \left(\int_\Gamma \int_D a_0(x) \nabla \ell(\cdot) \cdot \nabla \ell(\cdot)\, dx\, d\pi y + \sum_m \int_\Gamma y_m \int_D a_m(x) \nabla \ell(\cdot) \cdot \nabla \ell(\cdot)\, dx\, d\pi y \right)^{1/2}$$

$$= \left(\int_D a_0 \nabla \ell(\cdot) \cdot \nabla \ell(\cdot)\, dx \underbrace{\int_\Gamma d\pi y}_{=1} + \sum_m \int_{\Gamma_1} 1 \cdots \underbrace{\int_{\Gamma_m} y_m \cdots}_{=0} \int_{\Gamma_2} 1 \cdots \int_D a_m \nabla \ell(\cdot) \cdot \nabla \ell(\cdot) \right)^{1/2}$$

$$= \| a_0^{1/2}(x) \nabla \ell(g^*_x - g_x) \|^2_{L^2(D)}$$

$$\leq \underbrace{a_0^{max}}_{m_0} \| \nabla \ell(g^*_x - g_x) \|^2_{L^2(D)} \quad \text{with } m_0 := \left(\max_{x \in D} a_0(x) \right)^{1/2} \left(= 1 \text{ in our case} \right)$$

$$\overset{(*)}{\leq} m_0 \tilde{C} \underbrace{\| g^*_x - g_x \|_{H^{1/2}(\partial D)}}$$

$$\overset{(*_1)}{\leq} m_0 \tilde{C} \| h_x^{1/2} (g^*_x - g_x)' \|_{L^2(\partial D)} \,.$$

(*) as in eq. shown before eq (41) in FPP paper

(*1) from Lemma 1 of FPP paper since g_x is ALSO a nodal interpolant of g^*_x

Therefore squaring we obtain:

$$\| U^*_{xp} - U^*_{xp} \|^2_B \leq \underbrace{(m_0 \tilde{C})^2}_{=: m_0 \tilde{C}} \| h_x^{1/2} (g^*_x - g_x)' \|^2_{L^2(\partial D)} = m_0 \tilde{C} \| h_x^{1/2} (g^* - g)' \|^2_{L^2(\partial D)} \qquad (31)$$

Finally using (31) in (26) gives:

$$2|B(U-U_{xp}^*, U_{xp}^*-U_{xp})| \leq \alpha \|U-U_{xp}^*\|_B^2 + \alpha^{-1} m_0 \tilde{C} \|h_x^{1/2}(g_x^*-g_x)^1\|_{L^2(\partial D)}^2 \quad (32)$$

With (32) at hand now we can do the following.
Recall that $e^* = U_{xp}^* - U_{xp} \in V_{xp}^{g_x^*-g_x}$, $e = U-U_{xp} \in V^{g-g_x}$ and $U-U_{xp}^* \in V^{g-g_x^*}$. We have:

$$\|e\|_B^2 = \|U-U_{xp}^* + U_{xp}^* - U_{xp}\|_B^2 =$$
$$= \|U-U_{xp}^*\|_B^2 + \|U_{xp}^*-U_{xp}\|_B^2 + 2B(U-U_{xp}^*, U_{xp}^*-U_{xp}) \quad (33)$$

Use now the (lower) bound (32) in (33):

$$\|e\|_B^2 \geq \|U-U_{xp}^*\|_B^2 + \|e^*\|_B^2 - \alpha\|U-U_{xp}^*\|_B^2 - \alpha^{-1} m_0 \tilde{C} \|h_x^{1/2}(g_x^*-g_x)^1\|_{L^2(\partial D)}^2$$
$$= (1-\alpha)\|U-U_{xp}^*\|_B^2 + \|e^*\|_B^2 - \alpha^{-1} m_0 \tilde{C} \|\cdot\|_{L^2(\partial D)}^2$$

$$\Rightarrow \|e\|_B^2 + \alpha^{-1} m_0 \tilde{C} \|\cdot\|_{L^2(\partial D)}^2 \geq (1-\alpha)\|U-U_{xp}^*\|_B^2 + \|e^*\|_B^2 \geq \|e^*\|_B^2 \quad (34)$$

if we assume that $\boxed{\alpha < 1}$.

Now use the (upper) bound (32) in (33):

$$\|e\|_B^2 \leq \|U-U_{xp}^*\|_B^2 + \|e^*\|_B^2 + \alpha\|U-U_{xp}^*\|_B^2 + \alpha^{-1} m_0 \tilde{e} \|\cdot\|_{L^2(\partial D)}^2$$
$$= (1+\alpha)\|U-U_{xp}^*\|_B^2 + \|e^*\|_B^2 + \alpha^{-1} m_0 \tilde{C} \|\cdot\|_{L^2(\partial D)}^2$$
$$\overset{(S.A.)}{\leq} (1+\alpha)\beta^2 \|\underbrace{U-U_{xp}}_{=e}\|_B^2 + \|e^*\|_B^2 + \alpha^{-1} m_0 \tilde{C} \|\cdot\|_{L^2(\partial D)}^2$$

$$\Rightarrow (1-\beta^2(1+\alpha))\|e\|_B^2 \leq \|e^*\|_B^2 + \alpha^{-1} m_0 \tilde{C} \|\cdot\|_{L^2(\partial D)}^2$$

which makes sense only if $(1-\beta^2(1+\alpha))>0 \Rightarrow \boxed{\alpha < \frac{1}{\beta^2}-1}$.

In such case we have:

$$\|e\|_B \leq \frac{1}{\sqrt{1-\beta^2(1+\alpha)}} \left(\|e^*\|_B^2 + \alpha^{-1} m_0 \tilde{C} \| h_x^{1/2} (g_x^+ - g_x)' \|^2_{L^2(\partial D)} \right)^{1/2} \quad (35)$$

Putting (34) and (35) together we have

$$\text{(reliability)} \quad \|e\|_B \leq \frac{1}{\sqrt{1-\beta^2(1+\alpha)}} \left(\|e^*\|_B^2 + \alpha^{-1} m_0 \tilde{C} \| h_x^{1/2} (g_x^* - g_x)' \|^2_{L^2(\partial D)} \right)^{1/2} \quad (36)$$

$$\text{(efficiency)} \quad \|e^*\|_B \leq \left(\|e\|_B^2 + \alpha^{-1} m_0 \tilde{e} \| h_x^{1/2} (g_x^* - g_x) \|^2_{L^2(\partial D)} \right)^{1/2} \quad (37)$$

→ ALSO we should connect this with $g - g_x$ anyway...

with respectively $\alpha < \frac{1}{\beta^2}-1$ and $\alpha < 1$.

Note that according to what β is (unknown) the value for α "makes sause" anyway. In fact suppose that $\beta \simeq 0,9$: this means that $\alpha < \frac{1}{(0.9)^2} - 1 \simeq 0.23$ and then we also have $\alpha < 1$ automatically. If $\beta \simeq 0,5$ (unprobable case) we have $\alpha < \frac{1}{(0.5)^2} - 1 \equiv 3$ and then again we can pick $\alpha < 1$ ...

However I believe that (36)-(37) is rubbish. Above all (37). If I rewrite it as

$$\|e^*\|_B^2 - \alpha^{-1} m_0 \tilde{C} \| h_x^{1/2} (g_x^* - g_x) \|^2_{L^2(\partial D)} \leq \|e\|_B^2$$

is it guaranteed that this lhs is positive?

2° Error Bound

Given eq. (20) for e^*, we should consider then a different bilinear form B_0.

Take B_0 and let's e_0^* the solution of (20) when B_0 is used on the LHS:

$$B_0(e_0^*, v) = F(v) - B(U_{xp}, v) \quad \forall v \in V_{xp}^{0*} \quad (38)$$

↑ where the 0 here simply means that B_0 has been used.

As before (just to remark) write $e_0^* \in V_{xp}^{g_x^* - g_x}$ as $e_0^* = e_0^{0*} + e_0^{*g}$ with $e_0^{0*} \in V_{xp}^{0*}$ and $e_0^{*g} \in V_{xp}^{g_x^* - g_x}$. Then (38) can be rewritten as:

$$B_0(e_0^{0*}, v) = F(v) - B(U_{xp}, v) - B_0(e_0^{*g}, v) \quad \forall\, v \in V_{xp}^{0*} \tag{39}$$

In order to prove that e^* from (20) and e_0^* from (38) are such that $\|e^*\|_B \simeq \|e_0^*\|_{B_0}$ we do the following.

From (20) and (38) we observe that

$$B(e^*, v) = B_0(e_0^*, v) \quad \forall\, v \in V_{xp}^{*0} \tag{40}$$

Remind that $e^*, e_0^* \in V_{xp}^{g_x^* - g_x}$. We choose then $v = e^* - e_0^* \in V_{xp}^{0*}$.

$$B(e^*, e^* - e_0^*) = B_0(e_0^*, e^* - e_0^*) \Rightarrow$$

$$\|e^*\|_B^2 - B(e^*, e_0^*) = B_0(e_0^*, e^*) - \|e_0^*\|_{B_0}^2 \Rightarrow$$

$$\|e^*\|_B^2 + \|e_0^*\|_{B_0}^2 = B_0(e_0^*, e^*) + B(e^*, e_0^*)$$

$$\overset{(C.S)}{\leq} \|e_0^*\|_{B_0} \underbrace{\|e^*\|_{B_0}} + \|e^*\|_B \underbrace{\|e_0^*\|_B}$$

$$\leq \|e_0^*\|_{B_0} \sqrt{\Lambda}\, \|e^*\|_B + \|e^*\|_B \frac{1}{\sqrt{\lambda}} \|e_0^*\|_{B_0} \tag{41}$$

Then if we first see that:

$$\|e^*\|_B^2 \leq \|e^*\|_B^2 + \|e_0^*\|_{B_0}^2 \leq \|e_0^*\|_{B_0} \sqrt{\Lambda}\, \|e^*\|_B + \|e^*\|_B \frac{1}{\sqrt{\lambda}} \|e_0^*\|_{B_0}$$

$$\Rightarrow \quad \|e^*\|_B \leq \left(\sqrt{\Lambda} + \frac{1}{\sqrt{\lambda}}\right) \|e_0^*\|_{B_0} \tag{42}$$

In the same way:

$$\|e_0^*\|_{B_0}^2 \leq \|e^*\|_B^2 + \|e_0^*\|_{B_0}^2 \leq \|e_0^*\|_{B_0}\sqrt{\Lambda}\,\|e^*\|_B + \|e^*\|_B \frac{1}{\sqrt{\lambda}}\|e_0^*\|_{B_0}$$

$$\Rightarrow \quad \|e_0^*\|_{B_0} \leq \left(\sqrt{\Lambda} + 1/\sqrt{\lambda}\right)\|e^*\|_B. \qquad (43)$$

Then (42) and (43) together give

Again, it seems rubbish to me

$$\boxed{\left(\sqrt{\Lambda} + 1/\sqrt{\lambda}\right)^{-1}\|e_0^*\|_{B_0} \leq \|e^*\|_B \leq \left(\sqrt{\Lambda} + 1/\sqrt{\lambda}\right)\|e_0^*\|_{B_0} \qquad (44)}$$

$$\|v\|_{B_0}^2 = B_0(v,v) = \int_\Gamma \int_D a_0(x)\, \nabla v(x,y)\cdot \nabla v(x,y)\, dx\, d\pi y$$

$$= \int_\Gamma \langle A_0 v(y), v(y)\rangle\, d\pi y$$

$$\geq \alpha_0^{min} \int_\Gamma \|v(y)\|^2_{H_0^1(D)}\, d\pi y = \alpha_0^{min} \|v\|_V^2$$

Equivalence between $\|\cdot\|_{B_0}$ and $\|\cdot\|_B$ norms

$$\|v\|_{B_0}^2 = B_0(v,v) = \int_\Gamma \int_D a_0(x)\nabla v(x,y)\cdot\nabla v(x,y)\, dx\, d\pi y$$

$$\leq \int_\Gamma \langle A_0 V(y), V(y)\rangle\, d\pi(y)$$

$$\leq \alpha_0^{max} \int_\Gamma \|v(y)\|^2_{H_0^1(D)}\, d\pi(y) = \alpha_0^{max}\|V\|_V^2$$

$V = H_0^1(D)\otimes L^2_\pi(\Gamma)$ (*)

$$\Rightarrow \quad \alpha_0^{min}\|v\|_V^2 \leq \|v\|_{B_0}^2 \leq \alpha_0^{max}\|v\|_V^2 \qquad (1)$$

Also we know that $|B(v,v)| \leq \alpha_{max}\|v\|_V\|v\|_V$ and $B(v,v) \geq \alpha_{min}\|v\|_V^2$ that is: $\|v\|_B^2 \leq \alpha_{max}\|v\|_B^2$ and $\|v\|_B^2 \geq \alpha_{min}\|v\|_B^2$ which gives:

$$\alpha_{min}\|v\|_V^2 \leq \|v\|_B^2 \leq \alpha_{max}\|v\|_V^2 \qquad (2)$$

From (1) and (2) we have

$$\alpha_0^{min}\frac{\|v\|_B^2}{\alpha_{max}} \overset{(2)}{\leq} \underbrace{\alpha_0^{min}\|v\|_V^2 \leq \|v\|_{B_0}^2 \leq \alpha_0^{max}\|v\|_V^2}_{(1)} \overset{(2)}{\leq} \alpha_0^{max}\frac{\|v\|_B^2}{\alpha_{min}}$$

Let $\alpha_0^{min}/\alpha_{max} =: \lambda$ and $\alpha_0^{max}/\alpha_{min} =: \Lambda$ then we have

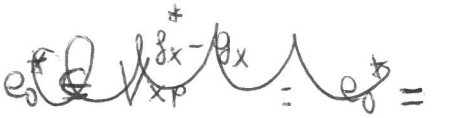
$$\boxed{\lambda\|v\|_B^2 \leq \|v\|_{B_0}^2 \leq \Lambda\|v\|_B^2} \qquad (3)$$

If instead of $H_0^1(D)$ we have $H_g^1(D)$ for whatever g, what should change here! (*)

$$\eta^2 \doteq \| e_{\Sigma P} + e_{\Sigma Q} \|_{B_0}^2 = \| e_{\Sigma P} \|_{B_0}^2 + \| e_{\Sigma Q} \|_{B_0}^2 + \underbrace{2 B_0(e_{\Sigma P}, e_{\Sigma Q})}_{=0 \text{ ortog. of polyn.}}$$

$$\eta \doteq \left(\| e_{\Sigma P} \|_{B_0}^2 + \| e_{\Sigma Q} \|_{B_0}^2 \right)^{1/2}$$

$$B_0(e_{\Sigma P}, v) = F(V) - B(U_{\Sigma P}, v)$$



Remind:

$$e^* = u_{\Sigma P}^* - u_{\Sigma P} \quad \text{with} \quad u_{\Sigma P} \in V_{\Sigma P}^g \ (u_{\Sigma P}|_{\partial\Omega} = g_\Sigma) \quad \text{then } e^* \in V_{\Sigma P}^{g^*} - V_{\Sigma P}^g$$
$$u_{\Sigma P}^* \in V_{\Sigma P}^{g^*} \ (u_{\Sigma P}^*|_{\partial\Omega} = g_\Sigma^*) \qquad =: V_{\Sigma P}^{g^*-g_\Sigma}$$

e_0^* simply come from using B_0 in place of B. Then $e_0^* \in V_{\Sigma P}^{g^*-g_\Sigma}$

$$B_0(e_{\Sigma P}, v) = F(V) - B(U_{\Sigma P}, v) \quad \forall v \in V_{\Sigma P}^0 \quad (\text{but } e_{\Sigma P} \in V_{\Sigma P}^g) \qquad (0.1)$$
$$B_0(e_{\Sigma Q}, v) = F(V) - B(U_{\Sigma P}, v) \quad \forall v \in V_{\Sigma Q}^0 \quad (\text{and also } e_{\Sigma Q} \in V_{\Sigma Q}^0) \qquad (0.2)$$

From (22) which remind here: $B_0(e_0^*, v) = F(V) - B(U_{\Sigma P}, v) \ \forall v \in V_{\Sigma P}^{0*}$
we have that

$$B_0(e_0^*, v) = F(V) - B(U_{\Sigma P}, v) = 0 \quad \forall v \in V_{\Sigma P}^0 \subset V_{\Sigma P}^{0*} \quad (\text{due to G.O.}) \qquad (1)$$

Also since $V_{\Sigma P}^0 \subset V_{\Sigma P}^{0*}$ we have $\forall v_{\Sigma P} \in V_{\Sigma P}^0$

$$B_0(e_0^*, v_{\Sigma P}) \overset{(22)}{=} F(v_{\Sigma P}) - B(U_{\Sigma P}, v_{\Sigma P}) \overset{(0.1)}{=} B_0(e_{\Sigma P}, v_{\Sigma P}) \quad \forall v_{\Sigma P} \in V_{\Sigma P}^0 \qquad (2)$$

and since also $V_{\Sigma Q}^0 \subset V_{\Sigma P}^{*0}$ we have $\forall v_{\Sigma Q} \in V_{\Sigma Q}^0$

$$B_0(e_0^*, v_{\Sigma Q}) \overset{(22)}{=} F(v_{\Sigma Q}) - B(U_{\Sigma P}, v_{\Sigma Q}) \overset{(0.2)}{=} B_0(e_{\Sigma Q}, v_{\Sigma Q}) \quad \forall v_{\Sigma Q} \in V_{\Sigma Q}^0 \qquad (3)$$

Choosing $v_{\Sigma Q} = e_{\Sigma Q}$ in (3) gives:

$$B_0(e_0^*, e_{\Sigma Q}) = B_0(e_{\Sigma Q}, e_{\Sigma Q}) \Rightarrow B_0(e_0^* - e_{\Sigma Q}, e_{\Sigma Q}) = 0 \quad \forall e_{\Sigma Q} \in V_{\Sigma Q}^0 \qquad (4)$$

write $e_{\Gamma p} = e^{0}_{\Gamma p} + R^{\vartheta}_{\Gamma p} \in V^{\vartheta}_{\Gamma p}$ with $e_{\Gamma p} \in V^{0}_{\Gamma p}$ and $R^{\vartheta}_{\Gamma p} \in V^{\vartheta}_{\Gamma p}$

in (2) plug $w_{\Gamma p} = e_{\Gamma p} - R^{\vartheta}_{\Gamma p}$:

$$B_0(e_0^*, e_{\Gamma p} - R^{\vartheta}_{\Gamma p}) = B_0(e_{\Gamma p}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p}) \qquad (5)$$

Now let's consider:

$$B_0(e_0^*, e_{\Gamma p} - R^{\vartheta}_{\Gamma p} + e_{\Gamma\alpha}) =$$

$$B_0(e_0^*, e_{\Gamma p} - R^{\vartheta}_{\Gamma p} + e_{\Gamma\alpha}) \pm B_0(e_{\Gamma\alpha}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p} - e_{\Gamma\alpha})$$

$$= B_0(e_0^* - e_{\Gamma\alpha}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p} + e_{\Gamma\alpha}) + B_0(e_{\Gamma\alpha}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p} - e_{\Gamma\alpha})$$

$$\overset{(4)}{=} B_0(e_0^* - e_{\Gamma\alpha}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p}) + B_0(e_{\Gamma\alpha}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p} - e_{\Gamma\alpha})$$

$$= B_0(e_0^*, e_{\Gamma p} - R^{\vartheta}_{\Gamma p}) - B_0(e_{\Gamma\alpha}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p}) + B_0(e_{\Gamma\alpha}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p}) + B_0(e_{\Gamma\alpha}, e_{\Gamma\alpha})$$

$$= B_0(e_0^*, e_{\Gamma p} - R^{\vartheta}_{\Gamma p}) + B_0(e_{\Gamma\alpha}, e_{\Gamma\alpha})$$

$$\overset{(5)}{=} B_0(e_{\Gamma p}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p}) + B_0(e_{\Gamma\alpha}, e_{\Gamma\alpha}) \qquad (6)$$

Applying correctly a Sch. ineq. on the LHS

$$B_0(e_{\Gamma p}, e_{\Gamma p} - R^{\vartheta}_{\Gamma p}) + \|e_{\Gamma\alpha}\|^2_{B_0} = B_0(e_0^*, e_{\Gamma p} - R^{\vartheta}_{\Gamma p} + e_{\Gamma\alpha})$$

$$\geq \|e_{\Gamma p}\|^2_{B_0} - \|e_{\Gamma p}\|_{B_0} \|R^{\vartheta}_{\Gamma p}\|_{B_0} \qquad \leq \|e_0\|^*_{B_0} \|e_{\Gamma p} - R^{\vartheta}_{\Gamma p} + e_{\Gamma\alpha}\|_{B_0}$$