"D" stands for Dirichlet $\|$ g_ℓ nodal interpolant of g (boundary)

(1) $\quad \mathrm{OSC}_{D,\ell}(E)^2 := |E| \, \| (g - g_\ell)' \|^2_{L^2(E)} \quad \forall \; E \in \mathcal{E}_\ell^D$ the Dirichleet edges

(2) $\quad \mathrm{OSC}^2_{D,\ell} := \sum_{E \in \mathcal{E}_\ell^D} \mathrm{OSC}_{D,\ell}(E)^2 = \sum_{E \in \mathcal{E}_\ell^D} |E| \, \| (g - g_\ell)' \|^2_{L^2(E)} \qquad |E| =: h_E$

$$= \sum_{E \in \mathcal{E}_\ell^D} \| h_E^{1/2} (g - g_\ell)' \|^2_{L^2(E)}$$

(3) $\quad \| g - g_\ell \|_{H^{1/2}(\Gamma_D)} \leq C \, \mathrm{OSC}_{D,\ell} \equiv C \left(\sum_{E \in \mathcal{E}_\ell^D} \| h_E^{1/2} (g - g_\ell)' \|^2_{L^2(E)} \right)^{1/2}$

Since g_ℓ is Also a nodal interpolant of g, then (3) holds also with g_ℓ^* in place of g:

(4) $\quad \| g_\ell^* - g_\ell \|_{H^{1/2}(\Gamma_D)} \leq C \left(\sum_{E \in \mathcal{E}_\ell^D} \| h_E^{1/2} (g_\ell^* - g_\ell)' \|^2_{L^2(E)} \right)^{1/2}$

(Lemma [FPP14])

The derivative of the nodal interpolant g_ℓ' is the elementwise best approximations of the derivative by piecewise constants, i.e.

(5) $\quad \| (g - g_\ell)' \|_{L^2(E)} = \min_{c \in \mathbb{R}} \| g' - c \|_{L^2(E)} \qquad E \in \mathcal{E}_\ell^D \qquad$ (32)[FPP14])

This implies that (Pythagoras)

(6) $\quad \| (g - \tilde{g}_\ell)' \|^2_{L^2(E)} = \| (g - g_\ell)' \|^2_{L^2(E)} + \| (g_\ell - \tilde{g}_\ell)' \|^2_{L^2(E)} \quad \forall \; \tilde{g}_\ell \in S^1(\mathcal{E}_\ell^D)$

and all edges $E \in \mathcal{E}_\ell^D$ ((33) [FPP14]).
Now (5) - (6) hold if we replace $g_\ell \mapsto g_\ell^* \equiv g_{\ell+1}$ and $\tilde{g}_\ell \mapsto g_\ell$, i.e.

(7) $\quad \| (g - g_\ell^*)' \|_{L^2(E)} = \min_{c \in \mathbb{R}} \| g' - c \|_{L^2(E)} \quad \forall \; E \in \mathcal{E}_{\ell^*}^D =: \mathcal{T}_\ell^* |_{\Gamma_D}$

(8) $\quad \| (g - g_\ell)' \|^2_{L^2(E)} = \| (g - g_{\ell^*})' \|^2_{L^2(E)} + \| (g_\ell^* - g_\ell)' \|^2_{L^2(E)} \quad \forall \; g_\ell \in S^1(\mathcal{E}_\ell^D)$
$\forall \; E \in \mathcal{E}_{\ell^*}^D$

From (8) we deduce that

$$\underbrace{\sum_{E\in\mathcal{E}_{\ell^*}^D}}\ \| h_E^{1/2}(g-g_\ell)'\|^2_{L^2(E)} = \sum_{E\in\mathcal{E}_{\ell^*}^D} \| h_E^{1/2}(g-g_\ell^*)'\|^2_{L^2(E)} + \sum_{E\in\mathcal{E}_{\ell^*}^D} \| h_E^{1/2}(g_\ell^*-g_\ell)'\|^2_{L^2(E)}$$

$$\Downarrow$$

$$\sum_{E\in\mathcal{E}_\ell^D} \| h_E^{1/2}(g-g_\ell)'\|^2_{L^2(E)} = \quad \| \quad + \quad \|$$

that is :

$$(9)\qquad \mathrm{OSC}^2_{D,\ell} = \mathrm{OSC}^2_{D,\ell^*} + \sum_{E\in\mathcal{E}_{\ell^*}^D} \| h_E^{1/2}(g_\ell^*-g_\ell)'\|^2_{L^2(E)}$$

from which we obtain

$$(10)\qquad \mathrm{OSC}^2_{D,\ell^*} \le \mathrm{OSC}^2_{D,\ell}$$

$$(11)\qquad \sum_{E\in\mathcal{E}_{\ell^*}^D} \| h_E^{1/2}(g_\ell^*-g_\ell)'\|^2_{L^2(E)} \le \mathrm{OSC}^2_{D,\ell}$$

(*) Consider the description 1° Error Bound at pag -4-

Let's write an alternative (simpler) description which leads to the bounds (36) - (37) for $\| U - U_{xp} \|_B \simeq \| U^*_{xp} - U_{xp} \|_B$.
In particular we avoid the use of the "artificial" solution U^*_{xp} belonging to V^{*g}_{xp} but having trace equal to g_x.

Let us start again from:

(1) $$\| U - U_{xp} \|^2_B = \| U - U^*_{xp} \|^2_B + \| U^*_{xp} - U_{xp} \|^2_B + 2B(U - U^*_{xp}, U^*_{xp} - U_{xp})$$

Now, consider the lift operator $\mathcal{L} : X_{\tilde{g}}|_{\partial D} \to X_{\tilde{g}}$ defined by

(2) $$\mathcal{L} w_{xp}|_{\partial D} := w_{xp} \quad \forall \; w_{xp} \in V^{\tilde{g}}_{xp} \qquad (\tilde{g} \text{ generic interpolant})$$

and with w_{xp} which is 0 in the interior nodes. So $\mathcal{L} g^*_x$ and $\mathcal{L} g_x$ are simple extension of the interpolants g^*_x and g_x to X_{g^*} and they are zero in the interior nodes (of T^*) ($\mathcal{L} g^*_x|_{\partial D} = g^*_x$, $\mathcal{L} g_x|_{\partial D} = g^0_x$)
In particular,

(3) $$\mathcal{L} g^*_x - \mathcal{L} g_x \in V^{*g}_{xp} \quad \text{and} \quad U^*_{xp} - U_{xp} - (\mathcal{L} g^*_x - \mathcal{L} g_x) \in V^{*0}_{xp}$$

since $(U^*_{xp} - U_{xp})|_{\partial D} = g^*_x - g_x$ and $(\mathcal{L} g^*_x - \mathcal{L} g_x)|_{\partial D} = g^*_x - g_x$.
Therefore, using the (G.O.) on eq. (24) we have:

(4) $$B(U - U^*_{xp}, U^*_{xp} - U_{xp} - (\mathcal{L} g^*_x - \mathcal{L} g_x)) = 0$$

which implies

(5) $$B(U - U^*_{xp}, U^*_{xp} - U_{xp}) \equiv B(U - U^*_{xp}, \mathcal{L} g^*_x - \mathcal{L} g_x).$$

Therefore we rewrite (1) using (5):

$$(6)\quad \|U - U_{x\rho}\|_B^2 = \|U - U_{x\rho}^*\|_B^2 + \|U_{x\rho}^* - U_{x\rho}\|_B^2 + 2B\left(U - U_{x\rho}^*,\ \ell g_x^* - \ell g_x\right)$$

Now using Young's inequality:

$$(7)\quad 2\left|B\left(U - U_{x\rho}^*,\ \ell g_x^* - \ell g_x\right)\right| \le \alpha \|U - U_{x\rho}^*\|_B^2 + \alpha^{-1}\|\ell g_x^* - \ell g_x\|_B^2$$

for some $\alpha > 0$. Also:

$$\|\ell g_x^* - \ell g_x\|_B^2 = \qquad \left(\Leftarrow \text{NOTE THAT } \ell g_x(x) \text{ doesn't depend on } y!\right)$$

$$= \int_\Gamma \int_D a_0(x)\left(\nabla\ell - \nabla\ell\right)\cdot\left(\nabla\ell - \nabla\ell\right)dx\,d\pi y + \sum_m^\infty \int_\Gamma \gamma_m \int_D a_m(x)\left(\nabla\ell - \nabla\ell\right)\cdot\left(\nabla\ell - \nabla\ell\right)dx\,d\pi y$$

$$= \|a_0^{1/2}(\nabla\ell g_x^* - \nabla\ell g_x)\|_{L^2(D)}^2 + \sum_m \|a_m^{1/2}(\nabla\ell g_x^* - \nabla\ell g_x)\|_{L^2(D)}^2 \underbrace{\int_\Gamma \gamma_m\, d\pi y}_{=0}$$

$$\le a_0^{max}\|\nabla\ell g_x^* - \nabla\ell g_x\|_{L^2(D)}^2$$

Here it holds $\|\nabla\ell g_x^* - \nabla\ell g_x\|_{L^2(D)} \le \|\ell g_x^* - \ell g_x\|_{H^1(D)}$

$$\le C_1 \inf\left\{\|w\|_{H^1(\Omega)} : w \in H^1(D),\ w|_{\partial D} = g_x^* - g_x\right\}$$

$$(8)\qquad =: C_1\|g_x^* - g_x\|_{H^{1/2}(D)}$$

$$\le a_0^{max} C_1^2 \underbrace{\|g_x^* - g_x\|_{H^{1/2}(D)}^2}$$

$$(8)\quad \le a_0^{max} C_1^2 C_2 \sum_{E \in \mathcal{E}_{x^*}^D} \|h_E^{1/2}(g_x^* - g_x)'\|_{L^2(E)}^2 \qquad \text{and also}$$

$$(9)\quad \le a_0^{max} C_1^2 C_2 \underbrace{\sum_{E \in \mathcal{E}_{x^*}^D} \|h_E^{1/2}(g - g_x)'\|_{L^2(E)}^2} =: a_0^{max} C_1^2 C_2\, OSC_{D,\ell}^2$$

Then we obtained (from (7))

$$(10)\quad 2\left|B\left(U - U_{x\rho}^*,\ \ell g_x^* - \ell g_x\right)\right| \le \alpha\|U - U_{x\rho}^*\|_B^2 + \alpha^{-1} a_0^{max} C_1^2 C_2\, OSC_{D,\ell}^2$$

Come back to (6) and use the lower bound from (10):

$$\|U - U_{xp}\|_B^2 \geq (1-\alpha)\|U - U_{xp}^*\|_B^2 + \|U_{xp}^* - U_{xp}\|_B^2 - \alpha^{-1} a_0^{max} c_1^2 c_2 \, osc_{D,\ell}^2$$

$$(\alpha < 1)\ \Big| \geq \|U_{xp}^* - U_{xp}\|_B^2 - \alpha^{-1} a_0^{max} c_1^2 c_2 \, osc_{D,\ell}^2$$

giving

$$\|U_{xp}^* - U_{xp}\|^2 \leq \|U - U_{xp}\|_B^2 + \alpha^{-1} a_0^{max} c_1^2 c_2 \, osc_{D,\ell}^2 \tag{11}$$

that it holds assuming $\alpha < 1$.