$\boxed{Au = f}$ $\quad v \approx u$, v approximation of $u \Rightarrow Av \approx f$

$\underbrace{e := u - v}_{\text{error}}$ and $\underbrace{r := Av - f = f - Av}_{\text{residual}}$

$Ae = \underbrace{Au} - Av = f - Av = r \Rightarrow \boxed{Ae = r}$ residual equation

$\boxed{u = e + v}$ residual correction

General Iterative Methods

$Au = f \Rightarrow$ SPLITTING of A: $A = P - N$ with P non-singular

general scheme:

$$Au = f \Rightarrow (P - N)u = f \Rightarrow Pu = Nu + f \Rightarrow u = P^{-1}Nu + P^{-1}f$$

$$\Rightarrow \boxed{u^{(k+1)} = P^{-1}Nu^{(k)} + P^{-1}f, \quad k \geq 0} \quad (1)$$

We will call $P^{-1}N =: B$ and $\tilde{f} = P^{-1}f$, therefore

$$\boxed{u^{(k+1)} = Bu^{(k)} + \tilde{f}, \quad k \geq 0} \quad (2)$$

Note that (2) can be written as we in another way. We firstly note that

$$A = P - N \Rightarrow P^{-1}A = Id - \underbrace{P^{-1}N} = Id - B \Rightarrow \underline{B = Id - P^{-1}A} \quad (3)$$

hence in (2) we can write

$$u^{(k+1)} = Bu^{(k)} + \tilde{f} = (Id - P^{-1}A)u^{(k)} + \tilde{f} = u^{(k)} + \underbrace{\tilde{f}} - P^{-1}Au^{(k)}$$

$$= u^{(k)} + P^{-1}f - P^{-1}Au^{(k)} = u^{(k)} + P^{-1}\underbrace{(f - Au^{(k)})}$$

$$= u^{(k)} + P^{-1}r^{(k)}$$

$$\Rightarrow \boxed{u^{(k+1)} = u^{(k)} + P^{-1}r^{(k)}}, \text{ with } r^{(k)} := f - Au^{(k)}, \quad k \geq 0 \quad (4)$$

where $r^{(k)}$ is then the residual at k^{th} iteration.

$$u^{(k+1)} = Bu^{(k)} + \tilde{f} = Bu^{(k)} + P^{-1}f$$

$$u^{(k+1)} = u^{(k)} + P^{-1}r^{(k)}, \quad r^{(k)} = f - Au^{(k)}$$

$$A = P - N = (D-E) - F \begin{cases} P = D-E \\ N = F \end{cases}; \quad B_{GS} = (D-E)^{-1}F$$

$D = \text{diag}(A)$
$E = -\text{tri}_{\text{low}}(A)$
$F = -\text{tri}_{\text{upp}}(A)$

$$u^{(k+1)} = (D-E)^{-1}F\,u^{(k)} + (D-E)^{-1}f \qquad \text{Gauss-Seidel}$$

$$u^{(k+1)} = u^{(k)} + (D-E)^{-1}(f - Au^{(k)}) = (D-E)^{-1}f + (Id - (D-E)^{-1}A)\,u^{(k)}$$

$e_h := u - u_h$
$r_h = b - Au_h = Ae_h$

Multigrid Method - V-Cycle

Three ① For an initial guess solve $A_h u_h = b$ and find u_h (say 3 GS or Jacobi steps)
the residual $r_h = b - Au_h \equiv Ae_h$ $\quad \underbrace{RAe_h}$

V-Cycle on 2-level h and 2h

② Restriction of the residue $r_h \to r_{2h}$ by $Rr_h = r_{2h}$ (R = restriction matrix (operator I_h^{2h}))

③ Now solve $A_{2h} E_{2h} = r_{2h}$, where r_{2h} is given in step ②
and $A_{2h} = RA_hI$, where now I is "Prolongation matrix)
or interpolation operator I_{2h}^{h})
and then get E_{2h} (say by using again 5 GS/Jacobi step with initial guess $e_{2h}=0$)

④ Interpolate the coarse error $E_{2h} \to E_h$ by $E_h = IE_{2h}$

5-last With the error E_h correct the solution u_h computed in step ①
by $u_h \leftarrow u_h + E_h \approx u$

Step ① = SMOOTHER ; Step ⑤ = POST-SMOOTHER

Note $E_h \approx e_h$ but they are not the same

$$e^{(k)} = u - u^{(k)} = u - Bu^{(k-1)} - P^{-1}f$$
$$e^{(k+1)} = u - u^{(k+1)} = u - Bu^{(k)} - P^{-1}f$$

$e^{(k)} = U - U^{(k)}$

$U^{(k+1)} = BU^{(k)} + \tilde{f}$ $\rightarrow$ $U - U^{(k+1)} = U - BU^{(k)} - \tilde{f}$

$U = BU + \tilde{f}$ $\rightsquigarrow$ $e^{(k)} = U^{(k)} - U = U^{(k)} - BU - \tilde{f}$

$= U - BU^{(k)} + P^{-1}\tilde{f}$

$B = P^{-1}N = Id - P^{-1}A \Rightarrow PA = Id - B \Rightarrow \boxed{P^{-1} = (Id - B)A^{-1}}$ (5)

Therefore: $e^{(k+1)} = U - U^{(k+1)} = U - BU^{(k)} + (Id - B)A^{-1}\tilde{f}$

$= U - BU^{(k)} + (Id - B)U$

$= (Id - Id + B)U - BU^{(k)}$

$= BU - BU^{(k)} = B(U - U^{(k)}) = Be^{(k)}$

Iterate:

$e^{(k+1)} = Be^{(k)} = BBe^{(k-1)} = BBBe^{(k-2)} = \dots$

we get

$e^{(k+1)} = B^{(k+1)} e^{(0)} \rightsquigarrow \boxed{e^{(k)} = B^k e^{(0)}, \quad k = 0, 1, 2, \dots}$ (6)

General error iteration (Iterative Method)

$e^{(k)} = B^k e^{(0)} = (Id - P^{-1}A)^k e^{(0)}, \quad k = 0, 1, 2 \dots$ (7)

V-Cycle: 2-Grid Iteration:

Repeat the loop for $i = 0, 1, \dots$ until convergences (Pag 93 Sylvester)

(1) Choose an initial guess $U_h^{(0)}$

(1) (Smoothing) Solve $A_h U_h = \tilde{f}$ by k step of GS/Jacobi Method:

$U_h \leftarrow (Id - P^{-1}A_h)U_h + P^{-1}\tilde{f}$

and get the residual $r_h = \tilde{f} - AU_h = Ae_h$

(2) Restrict the residual $r_h \rightarrow r_{2h}$ by $r_{2h} = R r_h = RAe_h$

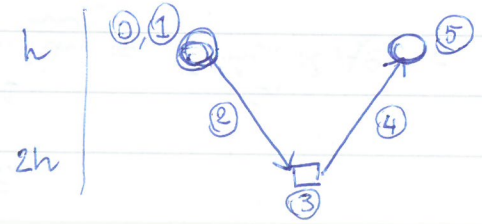
(3) Now solve exactly on coarse grid $A_{2h} E_{2h} = r_{2h}$

where $A_{2h} = RA_h I$ and r_{2h} is given in (2)

(4) (Prolongation) Interpolate $E_{2h} \rightarrow E_h$ by $E_h = IE_{2h}$

(5) (Post-Smoothing/Correction) Update the solution $U_h \leftarrow U_h + E_h$

This would be then the starting vector for new iteration in step (1)



Note the steps to get E_h

$$E_h = I\,E_{2h} = I\,A_{2h}^{-1} r_{2h} = I\,\underbrace{A_{2h}^{-1}}_{(I(RA_hI))^{-1}RA_h} e_{2h} = I(RA_hI)^{-1}RA_h e_h \doteq S e_h \quad (8)$$

Where $\boxed{S \doteq I(RA_hI)^{-1}RA_h}$ (9) matrix. This matrix satisfies

$$S^2 = I(RA_hI)^{-1}RA_h\; I\;(RA_hI)^{-1}RA_h = I(RA_hI)^{-1}RA_h \equiv S \Rightarrow \boxed{S^2 \equiv S} \quad (*)$$

(I between the two factors)

therefore $S^{(k)} \equiv S$, for every $k \geq 0$. This means in particular that

$S^2 e = Se$ ← consider $\{\lambda, e\}$ eigenpair of S

$S^2 e = S\,Se = S(\lambda e) = \lambda Se = \lambda^2 e$

but $S^2 \equiv S \Rightarrow Se \equiv S^2 e = \lambda^2 e$!

Therefore $\lambda^2 \equiv \lambda$ and the only eigenvalues possibles are just $\lambda = 0$ or $\lambda = 1$ (10)

$e^{(k)} = U - U^{(k)}$, $U_h^{(k)} \leftarrow U_h^{(k)} + E_h^{(k)}$ that is

$e^{(k)} = U - \ldots$

$$(A_h^{-1} - I A_{2h}^{-1} R) A_h (I - P A_h)^k e^{(i)}$$

$$(I - P^{-1}A_h)^k e^{(i)} - I A_{2h}^{-1} R A_h (I - P^{-1}A_h)^k e^{(i)}$$

$$- I(RA_hI)^{-1} RA_h (I - P^{-1}A_h)^k e^{(i)}$$

$e^{(i+1)} = U - U^{(i+1)}$

$e^{(i)} \leftarrow U_h^{(i)} + E_h^{(i)}$

the first term $U_h^{(i)} =$

$$P\bar{e} = P(\bar{A})^{-1}\bar{F} = P\bar{A}^{-1} \ldots RAe^{(i)}$$

($P\bar{A}^{-1} \to I(RA_hI)^{-1}$; $RAe^{(i)} \to (I - P^{-1}A_h)^k e^{(i)}$)

$\equiv$ error smoothing + error post-processed

$e^{(i+1)}_{smooth.}$ $\qquad$ $e^{(i+1)}_{p.p.}$

$$\equiv B^k e_h^{(i)} + E_{2h} = I(RA_hI)^{-1}RA\,e_h^{(i)}$$

($e_h^{(i)}$: error smoothing)

$$= B^k e^{(i)} + I(RAI)^{-1}RA\,B^k e^{(i)}$$

$$= (Id + I(RAI)^{-1}RA)\,B^k e^{(i)}$$

$$= (A^{-1} + I(RAI)^{-1}R)\,AB^k e^{(i)}$$

4

$e^{(i)} = U - U^{(i)}$

$U^{(i)} = \text{k step of } \left[(Id - P^{-1}A) U^{(i)} + P^{-1} f \right] + \underbrace{I(RAI)^{-1} RA}_{S} \underbrace{e_n}_{U - U^{(k),i}}$

$\Downarrow$ smoothing

$e^{(k)} = U - U^{(k)}$

error given by $Ae_n = r_n = f - A U_n^{(k)}$ where $U^{(k)}$ come from the smoothing

$= U^{(k,i)} + S(U - U^{(k)}) = (Id - S) U^{(k),i} + SU$

$\Rightarrow e^{(i)} = U - (Id - S) U^{(k),i} - SU = (Id - S) U - (Id - S) U^{(k),i} = (Id - S)(U - U^{(k),i})$

e^k error due to GS/Jacobi after k iterat.

$B^k (U - U_0^{(k)}) = B^k e_0^{(k)} = \leftarrow$

$e^{(i+1)} = (Id - S)(U - U^{(k),i+1}) = (Id - S) B^k (U - \underbrace{U_0^{(k),i+1}}_{\text{this is the initial guess of iteration } i+1 \text{ is the final } U^{(k),i} \text{ of the previous iteration}})$

$= (Id - S) B^k (U - (Id - S) U^{(k),i} - SU)$

$= (Id - S) B^k \underbrace{((Id - S)(U - U^{(k),i}))}_{e^{(i)}} = (Id - S) B^k e^{(i)}$

Therefore
$$\boxed{\begin{aligned} e^{(i+1)} &= (Id - S) B^k e^{(i)} \\ &= (A^{-1} - I(RAI)^{-1} R) A B^k e^{(i)} \\ &= (A^{-1} - I(RAI)^{-1} R) A \cdot (Id - P^{-1}A)^k e^{(i)} \end{aligned}}$$

$S = I(RAI)^{-1} RA$

$Id - S = Id - I(RAI)^{-1} RA = A^{-1}A - \dots$

$= (A^{-1} - I(RAI)^{-1} RA) A$

Let's rewrite well this ... 5

turn page

Let $e^{(i)} \doteq U - U^{(i)}$ the error after the ith iteration of the 2grid V-cycle and $U^{(i)}$ being the final solution at the end of the iteration.

Now $U^{(i)}$ is given by the first approximation given by the smoothing (GS/Jacobi) plus the correction (post smoothing) term E_{2h} in the code. This is:

$$U^{(i)} = k \text{ steps of } \left((Id - P^{-1}A) U^{(i)} + P^{-1} f \right) + \text{correction: } \underbrace{I(RAI)^{-1}RA}_{S} \underbrace{e_h}_{\uparrow}$$

$U^{(i)}$: this is the initial guess at the beginning of k steps

① ②

- part ① produce the solution $U^{k,i}$ giving an error of $U - U^{k,i}$
- in part ② the term $e_h \equiv U - U^{k,i}$ is the error given by the smoother, (which solves $A e_h = r_h = f - A U^{k,i}$)

Therefore

$$U^{(i)} = U^{k,i} + S(U - U^{k,i}) = (Id - S) U^{k,i} + SU \qquad (11)$$

So the error $e^{(i)}$ is:

$$e^{(i)} = U - U^{(i)} = U - (Id - S) U^{k,i} - SU = (Id - S)(U - U^{k,i}) \qquad (12)$$

Moreover the term $U - U^{k,i}$, i.e. the error produced by the smoother after k iteration during step i, is nothing but

$$U - U^{k,i} = B^k (U - U_0^{k,i}) \qquad (13)$$

Where $U_0^{k,i}$ is the initial guess for the smoother at the end of the k step iteration during step ith. See eq (6) general error for GS/Jacobi iterations.

Thus:

$$e^{(i)} = (Id - S) B^k (U - U_0^{k,i})$$

with $U_0^{k,i}$ initial guess for the smoother during iteration ith.

Ok. At the end of ith, we then get the solution $U^{(i)}$ that is used as initial guess for the following smoothing operation for $(i+1)$th step.

If we the consider now the following error:

$$e^{(i+1)} = (Id - S)(U - U^{k,i+1}) = (Id - S)B^k(U - U_0^{k,i+1})$$

where $U_0^{k,i+1}$ is the initial guess for something of $(i+1)$th step.
But this is given by the previous $U^{(i)}$! From (11) we get

$$\begin{aligned} e^{(i+1)} &= (Id - S)B^k(U - U_0^{k,i+1}) \\ &= (Id - S)B^k(U - U^{(i)}) \\ &= (Id - S)B^k(U - (Id - S)U^{k,i} - SU) \\ &= (Id - S)B^k((Id - S)(U - U^{k,i})) \\ &= \text{using (12)} = (Id - S)B^k e^{(i)} \end{aligned}$$

Then we have:

$$\boxed{e^{(i+1)} = (Id - S)B^k e^{(i)}} \quad (14)$$

Remark (Matrices)

- $B = Id - P^{-1}A$
- $S = I(RAI)^{-1}RA \Rightarrow Id - S = A^{-1}A - I(RAI)^{-1}RA$
$= (A^{-1} - I(RAI)^{-1}R)A$

Then if we expand (14) we have:

$$\boxed{e^{(i+1)} = (A^{-1} - I(RAI)^{-1}R)A(Id - P^{-1}A)^k e^{(i)}} \quad (15)$$

Formula for the Error $e^{(i)} = U - U^{(i)}$ at iteration ith

$$r^{(k)} = Ae^{(k)} = AB^k e^{(0)} = A(Id - \underbrace{P^{-1}A}_{P-N})^k e^{(0)}$$

~~$Id - P^{-1}(P-N) = Id - Id + P^{-1}N$~~

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$k=2$

$$A(Id - P^{-1}A)^2 e^{(0)} = A(Id - P^{-1}A)(Id - P^{-1}A)e^{(0)} =$$

$$= (A - AP^{-1}A)(Id - P^{-1}A)e^{(0)} =$$

$$= (A - \boxed{AP^{-1}A} - \boxed{AP^{-1}A} + \boxed{AP^{-1}AP^{-1}A})e^{(0)}$$

$$= (Id - 2AP^{-1} + (AP^{-1})^2)A$$

$$= (Id - AP^{-1})^2 \boxed{Ae^{(0)}} = (Id - AP^{-1})^2 r^{(0)}$$

So $\boxed{A(Id - P^{-1}A)^k \equiv (Id - AP^{-1})^k A}$

$A(Id - P^{-1}A)^{k+1} = A(Id - P^{-1}A)^k(Id - P^{-1}A) = (Id - AP^{-1})^k A(Id - P^{-1}A)$

$(A - \boxed{AP^{-1}A})$
$(Id - AP^{-1})A$

Other property (Ex 2.12 pag 109 Sylvester book)

For a general (k) step we know that the residual due to the smoothing oper. satisfy:

$$r^{(k)} = Ae^{(k)} = A B^k e^{(0)} = A(Id - P^{-1}A)^k e^{(0)}.$$

We need to prove first the following (general) property

$$A(Id - P^{-1}A)^k = (Id - AP^{-1})^k A \quad (16)$$

Proof: By Induction: $k=1 \Rightarrow$ lhs $A(Id - P^{-1}A) = A - AP^{-1}A$
rhs $(Id - AP^{-1})A = A - AP^{-1}A$ ↖ they are the same.

Suppose it's true for a value k. Then we have:

$$A(Id - P^{-1}A)^{k+1} = \underbrace{A(Id - P^{-1}A)^k}_{\text{ipotesi induzione}}(Id - P^{-1}A)$$

$$\equiv (Id - AP^{-1})^k A(Id - P^{-1}A)$$

$$= (Id - AP^{-1})^k (A - AP^{-1}A) = (Id - AP^{-1})^k (Id - AP^{-1})A$$

$$= (Id - AP^{-1})^{k+1} A$$

So it's true. (16) holds □

Alright. We can rewrite the residual as:

$$r^{(k)} = A(Id - P^{-1}A)^k e^{(0)} \equiv (Id - AP^{-1})^k Ae^{(0)} \equiv (Id - AP^{-1})^k r^{(0)} \quad (17)$$

where the last equality holds since $Ae^{(0)} \equiv r^{(0)} = f - Au^{(0)}$ is the residual on the 0th iteration. With eq. (17), let now multiply by A eq (15)

$$Ae^{(i+1)} = A(A^{-1} - I(RAI)^{-1}R)\underbrace{A(Id - P^{-1}A)^k e^{(i)}}_{\leftarrow \text{use (16)}}$$

$$= A(A^{-1} - I(RAI)^{-1}R)(Id - AP^{-1})^k Ae^{(i)}$$

and being now $r^{(i+1)} = Ae^{(i+1)}$, $r^{(i)} = Ae^{(i)}$ being the residual at iterations (i+1)th, ith from V-cycle with 2 grids, then holds:

$$r^{(i+1)} = A(A^{-1} - I(RAI)^{-1}R)(Id - AP^{-1})^k r^{(i)} \quad (18)$$