We want to consider (assume) the Saturation Assumption in our parametric framework, that is $\exists\, \beta \in (0,1)$ s.t.

$$\| U - U^*_{xp} \|_B \leq \beta \| U - U_{xp} \|_B \qquad (1)$$

where $U^*_{xp} \in V^*_{xp} := V_{xp} \oplus V_{yp} \oplus V_{xq}$ is an enriched solution of $U_{xp} \in V_{xp}$.

We would like to consider the following:

$$\boxed{\| U - U_{x^*p^*} \|_B \leq \| U - U_{x^*p} \|_B \leq \beta_h \| U - U_{xp} \|_B} \qquad (2)$$

Where:

- $U_{x^*p^*} \in V^*_{xp}$ is a double enriched solution (by both mesh ref. and polynomial enrich.)
- $U_{x^*p} \in V^*_{xp} := V_{xp} \oplus V_{yp}$ is an enriched solution (by mesh ref)
- $\beta_h \in (0,1)$ is the constant coming from the Saturation Assumption in non-parametric framework.

Now, the first inequality in (2) can be shown easily by using Galerkin orthogonality. (Or, in other words, can be just considered as the Saturation assumption with $\beta = 1$ when the polynomial enrichment is done). So suppose $U \in H^1_0(D) \otimes L^2_\Pi(\Gamma)$ is the exact solution. Then, let $U_{x^*p}$ be the Galerkin approx. satisf. $B(U_{x^*p}, v) = F(v)$ $\forall\, v \in X^* \otimes \mathcal{P}_p$. Moreover, let $U_{x^*p^*}$ be the enriched Galerkin approx satisfying instead $B(U_{x^*p^*}, v) = F(v)$ $\forall\, v \in X^* \otimes \mathcal{P}_{p^*}$ where

$p^* = p \cup q$ ← with $q$ being a computable (finite) index set $\subset \mathbb{N}$.

Therefore we have:

$$\| U - U_{x^*p} \|^2_B = \| U - U_{x^*p^*} \|^2_B + \| U_{x^*p^*} - U_{x^*p} \|^2_B + \underbrace{2B(U - U_{x^*p^*},\, U_{x^*p^*} - U_{xp})}_{=0}$$

$$\Downarrow$$

$$\| U - U_{x^*p^*} \|^2_B = \| U - U_{x^*p} \|^2 - \underbrace{\| U_{x^*p^*} - U_{x^*p} \|^2_B}_{\geq 0}$$

Then, we can conclude that

$$\| U - U_{x^*p^*} \|_B \leq \| U - U_{x^*p} \|_B \qquad (3)$$

which is the left-inequality of (2).

In order to make the right-inequality of (2) reasonable, we do the following. (2)

Let's fix a sample $\hat{y} \in \Gamma$ and consider the (deterministic) problem

$$-\operatorname{div}\left(a(x,\hat{y})\,\nabla U(x,\hat{y})\right) = f(x), \quad x \in D \qquad (3)$$

For the weak formulation, we know there exists a unique $U_{\hat{y}}(x) \in H_0^1(D)$ s.t. $\hat{B}(U_{\hat{y}}(x), v) = F(v)\ \forall\, v \in H_0^1(D)$. Note that $\hat{B}$ indicates that it is the bilinear form of the "pure deterministic" problem.

Now, we have the two Galerkin approximations:

$$\exists!\ U_{\hat{y}}^{h}(x) \in X \ \text{ s.t. } \ \hat{B}(U_{\hat{y}}^{h}(x), v) = F(v)\ \forall\, v \in X \qquad (4)$$

$$\exists!\ U_{\hat{y}}^{h*}(x) \in X^{*} \ \text{ s.t. } \ \hat{B}(U_{\hat{y}}^{h*}(x), v) = F(v)\ \forall\, v \in X^{*} \qquad (5)$$

where $X^*$ is therefore the enriched FE space associated with a refinement of the triangulation $\mathcal{T}$ (associated to $X = X_{\mathcal{T}}$).

(*) At this point, we assume a deterministic Saturation Assumption holds for the Galerkin approx., i.e., $\exists\ \beta_h \in (0,1)$, $\beta_h = \beta_h(\hat{y})$, s.t.

$$\| U_{\hat{y}} - U_{\hat{y}}^{h*} \|_{\hat{B}} \leq \beta_h(\hat{y}) \| U_{\hat{y}} - U_{\hat{y}}^{h} \|_{\hat{B}} \qquad \forall\, \hat{y} \in \Gamma, \qquad (6)$$

and this hold for any fixed sample $\hat{y} \in \Gamma$. In explicit (6) is:

$$\int_D a_{\hat{y}}(x) \left|\nabla(U_{\hat{y}}(x) - U_{\hat{y}}^{h*}(x))\right|^2 dx \leq \beta_h(\hat{y}) \int_D a_{\hat{y}}(x) \left|\nabla(U_{\hat{y}}(x) - U_{\hat{y}}^{h}(x))\right|^2 dx \qquad (7)$$

for all $\hat{y} \in \Gamma$, and $a_{\hat{y}}(x)$ denoting the deterministic coefficient.

(*) If now we want to relax (7) and integrate over the whole space $\Gamma$, we have,

$$\int_\Gamma \int_D a(x,\hat{y}) \left|\nabla(U(x,\hat{y}) - U^{h*}(x,\hat{y}))\right|^2 dx\, d\pi(\hat{y}) \leq \int_\Gamma \beta_h(\hat{y}) \int_D a(x,\hat{y}) \left|\nabla(U(x,\hat{y}) - U^{h}(x,\hat{y}))\right|^2 dx\, d\pi(\hat{y}) \qquad (8)$$

where now in (8) everything depend also on $y$. Moreover, we can take the worse case for $\beta_h$-s and taking $\beta_h := \max_{\hat{y}\in\Gamma} \beta_h(\hat{y}) \in (0,1)$ that still would be in $(0,1)$. Thus:

$$\int_\Gamma \int_D a(x,y) \left|\nabla(U(x,y) - U^{h*}(x,y))\right|^2 dx\, d\pi_y \le \beta_h \int_\Gamma \int_D a(x,y) \left|\nabla(U(x,y) - U^{h}(x,y))\right|^2 dx\, d\pi_y \quad (9)$$

We get rid of "$\wedge$" in (9) since there's the $y$-dependency.

The "problem" with (9) is that we should ensure that $U^{h*}(x,y)$, $U^h(x,y)$ represent the Galerkin approximations of $U(x,y) \in H_0^1(D) \otimes L^2_\pi(\Gamma)$.

¿ QUESTION ? : In integrating (7) over $\Gamma$, is it true that $U^{*h}_{\hat{y}}(x) \in X^*$ and $U^h_{\hat{y}}(x) \in X$ become the Galerkin solutions $U^{*h}(x,y) \in X^* \otimes \mathcal{P}_p$ and $U^h(x,y) \in X \otimes \mathcal{P}_p$ approximating $U(x,y) \in H_0^1 \otimes L^2_\pi$ ?

---

To answer to ¿ QUESTION ?, let's consider the Stochastic Problem

$$B(U(x,y), v) = F(v) \qquad \forall\, v \in H_0^1 \otimes L^2_\pi \qquad (10) \qquad (\Leftarrow \exists!\, U(x,y))$$

and its Galerkin approximation

$$B(U_{XP}, v) = F(V) \qquad \forall\, v \in X^h \otimes \mathcal{P}_p \qquad (11) \qquad (\Leftarrow \exists!\, U_{XP})$$

which is in explicity

$$\int_\Gamma \int_D a(x,y)\, \nabla U_{XP}(x,y) \cdot \nabla v(x,y)\, dx\, d\pi(y) = \int_\Gamma \int_D f(x) v(x,y)\, dx\, d\pi(y), \quad v \in X^h \otimes \mathcal{P}_p \qquad (12)$$

Ok. Now consider the deterministic problem where we fix a sample $\hat{B}(U^h_{\hat{y}}(x), V) = F(v) \quad \forall\, v \in H_0^1(D)$ and the Galerkin approx (4) (see previous page. Write (4) explicitly:

$$\int_D a_{\hat{y}}(x)\, \nabla U^h_{\hat{y}}(x) \cdot \nabla v(x)\, dx = \int_D f(x) V(x)\, dx \quad \forall\, v \in X \qquad (13)$$

for all $\hat{y} \in \Gamma$.

Since in (13) we have $\forall v \in X$, in particular we can choose some test functions $v$, let's say, the "first P" of them, i.e. consider the set $\{v_\nu(x)\}$, $\nu \in P$, and each $v_\nu(x) \in X$.

Here, P is our index set of cardinality $\#P := P$. Therefore, we can write (13) as "a system" of P relations that would hold:

$$\left\{ \int_D a_{\hat{y}}(x) \nabla U^h_{\hat{y}}(x) \cdot \nabla v_{\nu(i)}(x)\, dx = \int_D f(x)\, v_{\nu(i)}(x)\, dx \quad \forall v_{\nu(i)} \in X \right. \tag{14}$$

with $i = 1 \dots P$ and the functions enumerated $v^{(1)} \dots v^{(P)}$.

Now, multiply the i-th equation in (14) by $P_{\nu(i)}(\hat{y}) \in \mathbb{R}$ which is a number since $\hat{y} \in \Gamma$ is fixed, then:

$$\left\{ \int_D a_{\hat{y}}(x) \nabla U^h_{\hat{y}}(x) \cdot \nabla v_{\nu(i)}(x)\, P_{\nu(i)}(\hat{y})\, dx = \int_D f(x)\, v_{\nu(i)}(x) P_{\nu(i)}(\hat{y})\, dx \right. \tag{15}$$

and summing over i: (and passing the sum inside the integrals):

$$\int_D a_{\hat{y}}(x) \nabla U^h_{\hat{y}}(x) \cdot \nabla \left( \sum_{\nu \in P} v_\nu(x) P_\nu(\hat{y}) \right) dx = \int_D f(x) \left( \sum_{\nu \in P} V_\nu(x) P_\nu(\hat{y}) \right) dx \tag{16}$$

Now define:

$$\sum_{\nu \in P} v_\nu(x) P_\nu(\hat{y}) =: v_{\hat{y}}(x) \in X \tag{17}$$

and it belongs to X since $\{P_\nu(\hat{y})\}$ are numbers. So rewrite (16) as:

$$\int_D a_{\hat{y}}(x) \nabla U^h_{\hat{y}}(x) \cdot \nabla v_{\hat{y}}(x)\, dx = \int_D f(x)\, v_{\hat{y}}(x)\, dx \tag{18}$$

(18) holds for any $\hat{y} \in \Gamma$ (and any $v_{\hat{y}} \in X$ ???)

If we now relax (18) and integrate over $\Gamma$ (getting rid of "$\wedge$") : (5)

$$\int_\Gamma \int_D a(x,y) \boxed{\nabla U^h(x,y)} \cdot \nabla v(x,y)\, dx\, d\pi(y) = \int_\Gamma \int_D f(x)\, v(x,y)\, dx\, d\pi(y) \qquad (19)$$

In (19) everything is now $y$-dependent and because of (17), the new $v(x,y)$ in (19) are test functions in $X \otimes \mathcal{P}_p$.

If we compare (19) with (12) we see that we almost get the same. Indeed, the Galerkin solution $U_{XP} \in X \otimes \mathcal{P}_p$ of (12) is the unique approximation of $U(x,y) \in H_0^1 \otimes L^2_\pi$ satysfying (10).

Then, this implies that $\nabla U^h(x,y)$ in (19) has to be $U_{XP}$!

This answer to ¿QUESTION? = it is true that when relaxing (integrating over $\Gamma$) the deterministic Galerkin approximation $U^h_y(x) \in X$ it becomes the Galerkin (stochastic) approxim. $U^h(x,y) \in X \otimes \mathcal{P}_p$ approximating the exact $U(x,y) \in H_0^1 \otimes L^2_\pi$ satisfying (10). Obviously, the same argument holds by consistency the "enriched" deterministic problem (5).

RECAP: the right-inequality of (2), i.e.,

$$\| U - U_{X^*_P} \|_B \leq \beta_h \| U - U_{XP} \|_B \qquad (20)$$

holds with $\beta_h := \max_{\hat{y} \in \Gamma} \beta_h(\hat{y}) \in (0,1)$ where each $\beta_h(\hat{y}) \in (0,1)$ is the constant appearing in (6) IF WE ASSUME that the DETERMINISTIC SATURATION ASSUMPTION HOLDS (for each sample $\hat{y} \in \Gamma$).