BRIEF NOTES ON THE SPARSE GRID STOCHASTIC COLLOCATION FINITE ELEMENT METHOD

For $M \in \mathbb{N}$, let $\Gamma^M := \bigotimes_{m=1}^{M} \Gamma^m$ with $\Gamma^m := [-1,1]$ for all $m \in \mathbb{N}$ be the truncated *parameter domain* and consider the space $H_0^1(D)$ for a domain $D \subset \mathbb{R}^2$ with polygonal domain ∂D.

1. One-dimensional interpolation

Let $i \in \mathbb{N}$ be an *index* or *level*. Let $Y^i := \{y_1^{(i)}, \dots y_{m_i}^{(i)}\} \subset [-1,1]$ be a set of $m_i \in \mathbb{N}$ interpolation (or collocation) points for (the univariate) Lagrange interpolation on $[-1,1]$.

Consider the case $M = 1$, i.e., $\Gamma^1 = [-1,1]$. Let $u \in C^0(\Gamma^1; H_0^1(D))$ and consider the one-dimensional Lagrange interpolation operators $\mathcal{U}^i : C^0(\Gamma^1; H_0^1(D)) \to V_{m_i}(\Gamma^1; H_0^1(D))$,

$$\mathcal{U}^i(u)(y) := \sum_{j=1}^{m_i} u(y_j^{(i)}) L_j^{(i)}(y) \qquad \forall\, y \in \Gamma^1, \tag{1}$$

where $L_j^{(i)} \in \mathbb{P}_{m_i-1}(\Gamma^1)$ are the Lagrange polynomials of degree $m_i - 1$ defined by

$$L_j^{(i)}(y) := \prod_{\substack{k=1 \\ k \neq j}}^{m_i} \frac{y - y_k^{(i)}}{y_j^{(i)} - y_k^{(i)}} \quad \text{such that} \quad L_j^{(i)}(y_k^{(i)}) = \delta_{jk}, \qquad \forall\, y \in \Gamma^1, \tag{2}$$

and where, for $m \in \mathbb{N}$,

$$V_m(\Gamma^1; H_0^1(D)) := \left\{ v \in C^0(\Gamma^1; H_0^1(D)) \ : \ v(x,y) = \sum_{k=1}^{m} z_k(x) L_k(y),\ \{z_k(x)\}_{k=1}^m \in H_0^1(D) \right\}.$$

Given the Lagrange operator (1), one easily obtains a quadrature formula. Suppose that we want to integrate $u \in C^0(\Gamma^1; H_0^1(D))$ on Γ^1. We can then approximate the integral by integrating its Lagrange interpolator, i.e.,

$$\int_{\Gamma^1} u(y)dy \approx \int_{\Gamma^1} \mathcal{U}^i(u)(y)dy = \sum_{j=1}^{m_i} u(y_j^{(i)}) \int_{\Gamma^1} L_j^{(i)}(y)dy = \sum_{j=1}^{m_i} u(y_j^{(i)}) w_j^{(i)} =: \mathcal{Q}^i(u)(y) \tag{3}$$

where $w_j^{(i)} := \int_{\Gamma^1} L_j^{(i)}(y)dy$, $j = 1, \dots, m_i$, are the *weights* of the quadrature formula $\mathcal{Q}^i$.

Notice that here, and in what follows, we hide the spatial dependency of u. For example, we simply wrote $u(y_j^{(i)})$ in (1) instead of $u(\boldsymbol{x}, y_j^{(i)})$.

2. Full tensor product interpolation

In the multivariate case $M > 1$, let $\boldsymbol{i} := (i_1, \dots, i_M) \in \mathbb{N}^M$ be a multi-index and consider the M sets of m_{i_k} interpolation points $Y^{i_k} := \{y_1^{(i_k)}, \dots, y_{m_{i_k}}^{(i_k)}\}$, $k = 1, \dots, M$. For $\boldsymbol{y} = (y_1, \dots, y_M) \in \Gamma^M$, and $u \in C^0(\Gamma^M; H_0^1(D))$, we define the *full-tensor interpolation formula*:

$$(\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_M})(u)(\boldsymbol{y}) := \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_M=1}^{m_{i_M}} u(y_{j_1}^{(i_1)}, \dots, y_{j_M}^{(i_M)}) \left(L_{j_1}^{(i_1)}(y_1) \otimes \cdots \otimes L_{j_M}^{(i_M)}(y_M) \right). \tag{4}$$

Clearly, the product above needs $\prod_{k=1}^{M} m_{i_k}$ function evaluations, i.e., formula (4) considers the full set of interpolation points given by the cartesian product of the sets Y^{i_k}, for $k = 1, \ldots, M$,

$$\mathcal{X}(M) := Y^{i_1} \times \cdots \times Y^{i_M}. \tag{5}$$

As expected, in most of the cases formula (4) is infeasible due to the number of points $\#\mathcal{X}(M)$ which grows quickly.

Similarly to the one-dimensional case, the associated full tensor product quadrature formula is given by

$$(\mathcal{Q}^{i_1} \otimes \cdots \otimes \mathcal{Q}^{i_M})(u)(\boldsymbol{y}) := \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_M=1}^{m_{i_M}} u(y_{j_1}^{(i_1)}, \ldots, y_{j_M}^{(i_M)}) w_{j_1}^{(i_1)} \cdots w_{j_M}^{(i_M)}, \tag{6}$$

where the weights $w_{j_k}^{(i_k)}$ are defined as the integral of the univariate j_k-th Lagrange polynomials on Γ_k,

$$w_{j_k}^{(i_k)} := \int_{\Gamma_k} L_{j_k}^{(i_k)}(y_k) dy_k \qquad \forall\, k = 1, \ldots, M. \tag{7}$$

3. Isotropic Smolyak interpolation

The Smolyak *isotropic* interpolation tries to reduce the overall number of function evaluations needed by the full tensor product formula (4) by defining conveniently a subset of the full cartesian product set $\mathcal{X}(M)$ without deteriorating the approximation error. In particular, the Smolyak formula is just a linear combination of product formulas (4) with the following key property: only products with a relatively small number of interpolation points is used.

In this section, we implicitly follow [NTW08b]. The construction proceeds as follows. Define the *difference operator*,

$$\mathcal{U}^0 := 0 \quad \text{and} \quad \Delta^i = \mathcal{U}^i - \mathcal{U}^{i-1} \quad \forall\, i \in \mathbb{N}, \tag{8}$$

where $\mathcal{U}^i$ is the one-dimensional Lagrange interpolator operator (1). Let $\boldsymbol{i} := (i_1, \ldots, i_M) \in \mathbb{N}^M$ be a multi-index with $|\boldsymbol{i}| := \sum_{n=1}^{M} i_n$ and $\ell \in \mathbb{N}_0$ be an integer defining the *level* of the interpolation. Define the following *index sets*:

$$X(\ell, M) := \left\{ \boldsymbol{i} \in \mathbb{N}^M \ : \ \sum_{n=1}^{M} (i_n - 1) \leq \ell \right\}, \tag{9}$$

$$\widetilde{X}(\ell, M) := \left\{ \boldsymbol{i} \in \mathbb{N}^M \ : \ \sum_{n=1}^{M} (i_n - 1) = \ell \right\}, \tag{10}$$

$$Y(\ell, M) := \left\{ \boldsymbol{i} \in \mathbb{N}^M \ : \ \ell - M + 1 \leq \sum_{n=1}^{M} (i_n - 1) \leq \ell \right\}. \tag{11}$$

Note that the multi-indices $\boldsymbol{i}$ in X belong to the unit simplex $|\boldsymbol{i}| \leq \ell + M$.

For a function $u \in C^0(\Gamma^M; H_0^1(D))$ and $\boldsymbol{y} \in \Gamma^M$, the *isotropic* Smolyak interpolator operator of level ℓ is given by

$$\mathcal{A}(\ell, M)(u)(\boldsymbol{y}) := \sum_{\boldsymbol{i} \in X(\ell, M)} \left(\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_M} \right)(u)(\boldsymbol{y}). \tag{12}$$

For all $\boldsymbol{i} \in \mathbb{N}^M$, the operator

$$\Delta^{\boldsymbol{i}} := \Delta^{i_1} \otimes \cdots \otimes \Delta^{i_M} = \bigotimes_{k=1}^{M} (\mathcal{U}^{i_k} - \mathcal{U}^{i_k - 1}), \tag{13}$$

is typically referred to as *hierarchical surplus* operator. Equivalently, formula (12) can written using the set $Y(\ell, M)$ as,

$$\mathcal{A}(\ell, M)(u)(\boldsymbol{y}) = \sum_{\boldsymbol{i} \in Y(\ell, M)} (-1)^{\ell + M - |\boldsymbol{i}|} \binom{M-1}{\ell + M - |\boldsymbol{i}|} \left(\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_M}\right)(u)(\boldsymbol{y}). \tag{14}$$

The set of interpolation, or collocation point in this case, needed to compute $\mathcal{A}(\ell, M)$ is given by

$$\mathcal{Y}(\ell, M) := \bigcup_{\boldsymbol{i} \in Y(\ell, M)} (Y^{i_1} \times \cdots \times Y^{i_M}) \ \subset \ \Gamma^M, \tag{15}$$

where we recall that $Y^i \subset \Gamma^m = [-1, 1]$ is the set of interpolation points used by the one-dimensional Lagrange interpolator operator $\mathcal{U}^i$ of level $i \in \mathbb{N}$. The set $\mathcal{Y}(\ell, M)$ is what is typically referred to as *sparse grid*.

Whenever the sets Y^i are nested, i.e., $Y^i \subset Y^{i+1}$, then $\mathcal{Y}(\ell, M) \subset \mathcal{Y}(\ell + 1, M)$ and

$$\mathcal{Y}(\ell, M) := \bigcup_{\boldsymbol{i} \in \widetilde{X}(\ell, M)} (Y^{i_1} \times \cdots \times Y^{i_M}). \tag{16}$$

By comparing (15) and (16), since $\widetilde{X} \subseteq Y$, notice that the Smolyak formula using nested points needs fewer functions evaluation then the formula using non-nested points.

The quadrature formula associated with the Smolyak interpolator operator $\mathcal{A}(\ell, M)$ is naturally defined as follows. Define

$$\mathcal{Q}^0 := 0 \quad \text{and} \quad \Delta_{\mathcal{Q}}^i := \mathcal{Q}^i - \mathcal{Q}^{i-1} \quad \forall\, i \in \mathbb{N}, \tag{17}$$

where $\mathcal{Q}^i$ denotes the one-dimensional quadrature formula (3) of level $i \in \mathbb{N}$. Then the Smolyak quadrature formula is given by

$$\mathcal{Q}(\ell, M)(u)(\boldsymbol{y}) := \sum_{\boldsymbol{i} \in X(\ell, M)} (\Delta_{\mathcal{Q}}^{i_1} \otimes \cdots \otimes \Delta_{\mathcal{Q}}^{i_M}). \tag{18}$$

For a more exhaustive overview of the Smolyak interpolation and quadrature see, e.g., [NR96, GG98, BNR00].

Remark 1. *The Smolyak interpolation formula* (12) *is said to be* isotropic *because "all M directions are treated equally". In practice, this means that if a multi-index $\boldsymbol{i} = (i_1, \ldots, i_M) \in X(\ell, M)$, then also any permutation of $\boldsymbol{i}$ belongs to $X(\ell, M)$ and contributes to the construction of $\mathcal{A}(\ell, M)$. Equivalently, the isotropic constructions uses the indices $\boldsymbol{i} \in X(\ell, M)$ which is the the unit simplex $\boldsymbol{i} \leq \ell + M$.*

Remark 2. *Let us remark the expression for the operator $\mathcal{A}(\ell, M) \in C^0(\Gamma^M; H_0^1(D))$ from either* (12) *or* (14)*, is a linear combination of the "basis functions" $\{\widetilde{L}_k\}$ and coefficients given*

by the values of u at the collocation points $\{\boldsymbol{y}_k\}$, for $k = 1, \dots, \#\mathcal{Y}(\ell, M) := Q$, i.e., compactly we can write:

$$\mathcal{A}(\ell, M)(u)(\boldsymbol{y}) = \sum_{k=1}^{Q} u(\boldsymbol{y}_k)\widetilde{L}_k(\boldsymbol{y}), \qquad \boldsymbol{y}_k \in \mathcal{Y}(\ell, M),\ \boldsymbol{y} \in \Gamma^M, \tag{19}$$

where then $Q \in \mathbb{N}$ denotes the cardinality of the sparse grid. The expression for each basis $\widetilde{L}_k$ is not immediately visible since it is a linear combination of the univariate Lagrange polynomials (2) *which has to be carried out according to the current level ℓ and dimension M under consideration. Furthermore, the set of functions $\{\widetilde{L}_k\}_{k=1}^{Q}$ has the delta property $\widetilde{L}_k(\boldsymbol{y}_s) = \delta_{ks}$ which is inherited from the delta property of the univariate Lagrange polynomials. In general the polynomial space these basis belong to is denoted as $\mathcal{P}_{\boldsymbol{p}}(\Gamma^M) \subset L^2_\pi(\Gamma^M)$ and defined as follows (see, e.g,* [NTW08b, pag. 2316]*). For $\boldsymbol{p} = (p_1, \dots, p_M) \in \mathbb{N}^M$,*

$$\mathcal{P}_{\boldsymbol{p}}(\Gamma^M) = \bigotimes_{m=1}^{M} \mathcal{P}_{p_m}(\Gamma^m) \quad \textit{with} \quad \mathcal{P}_{p_m}(\Gamma^m) := \operatorname{span}\{y_m^k \ :\ k = 0, \dots, p_m\}, \quad m = 1, \dots, M.$$

Analogously, the Smolyak quadrature formula (18) *can be written compactly as*

$$\mathcal{Q}(\ell, M) = \int_{\Gamma^M} \mathcal{A}(\ell, M)(u)(\boldsymbol{y})dy = \sum_{k=1}^{Q} u(\boldsymbol{y}_k) \int_{\Gamma^M} \widetilde{L}_k(\boldsymbol{y})dy = \sum_{k=1}^{Q} u(\boldsymbol{y}_k)\widetilde{w_k}, \tag{20}$$

where then $\widetilde{w_k} := \int_{\Gamma^M} \widetilde{L}_k(\boldsymbol{y})d\boldsymbol{y}$ are the associated weights. Similarly to the basis $\{\widetilde{L}_k\}$, these weights are linear combinations of the weights of the univariate quadrature formulas (see (3)*).*

In addition, with refer to the full-tensor product formula (4)*, let us stress that for a given collocation point $\boldsymbol{y}_k$ of the sparse grid $\mathcal{Y}(\ell, M)$, which also belongs to the full cartesian product set of collocation points, the corresponding "basis" $\widetilde{L}_k$ does not coincide (of course) with the associated "basis", say $\widehat{L}_k$, which would come from formula* (4)*.*

4. Anisotropic Smolyak interpolation

The Smolyak construction presented in the previous section is very effective if applied to functions whose dependency on the variables "weights" equally in all M directions of Γ^M. With a reference to the stochastic collocation FEM described later in section 7, the isotropic Smolyak construction does not seem a good choice if the random field of the problem depends on stochastic parameters which do not "weights" equally in the solution.

A generalization of the Smolyak sparse grid method considers a different index set which is defined by using a multi-index $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_M) \in \mathbb{R}_+^M$ that would be a *weight* factor for the different dimensions. Let $\underline{\boldsymbol{\alpha}} := \min_{1 \le m \le M} \alpha_m$ be the minumum of $\boldsymbol{\alpha}$. A more general class of simplices is defined by the following index set

$$X_{\boldsymbol{\alpha}}(\ell, M) := \left\{ \boldsymbol{i} \in \mathbb{N}^M \ :\ \sum_{n=1}^{M} \alpha_n(i_n - 1) \le \underline{\boldsymbol{\alpha}}\ell \right\}. \tag{21}$$

Then, analogously to the isotropic case, the *anisotropic* Smolyak interpolator operator of level ℓ is defined by

$$\mathcal{A}_{\boldsymbol{\alpha}}(\ell, M)(u)(\boldsymbol{y}) := \sum_{\boldsymbol{i} \in X_{\boldsymbol{\alpha}}(\ell, M)} \left(\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_M}\right)(u)(\boldsymbol{y}). \tag{22}$$

Equivalently, this can be rewritten as

$$\mathcal{A}_{\boldsymbol{\alpha}}(\ell, M) = \sum_{\boldsymbol{i} \in Y_{\boldsymbol{\alpha}}(\ell, M)} \sum_{\substack{\boldsymbol{j} \in \{0,1\}^M \\ \boldsymbol{i}+\boldsymbol{j} \in X_{\boldsymbol{\alpha}}(\ell, M)}} (-1)^{|\boldsymbol{j}|} \left(\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_M}\right)(u)(\boldsymbol{y}), \tag{23}$$

where $Y_{\boldsymbol{\alpha}}(\ell, M)$ is the index set

$$Y_{\boldsymbol{\alpha}}(\ell, M) := X_{\boldsymbol{\alpha}}(\ell, M) \setminus X_{\boldsymbol{\alpha}}\left(\ell - \frac{|\boldsymbol{\alpha}|}{\underline{\boldsymbol{\alpha}}}, M\right).$$

The sparse grid set of collocation points is then given by

$$\mathcal{Y}_{\boldsymbol{\alpha}}(\ell, M) := \bigcup_{\boldsymbol{i} \in Y_{\boldsymbol{\alpha}}(\ell, M)} (Y^{i_1} \times \cdots \times Y^{i_M}).$$

Note that the isotropic Smolyak formula of previous section is a particular case of the anisotropic construction made here. This can be easily verified by choosing $\boldsymbol{\alpha} = (1, \ldots, 1)$, i.e., by weighting equally all directions.

It is clear that in order to use the proposed strategy here, one has to choose the weight $\boldsymbol{\alpha} \in \mathbb{R}_+^M$ from either a *priori* or *posteriori* knowledge; see, e.g., [Web07, NTW08a].

5. Clenshaw-Curtis collocation points

The sparse grid Clenshaw-Curtis set of collocation points is one of the most used choice. Firstly, we choose the number of points $m_i \in \mathbb{N}$, $i \in \mathbb{N}$, in each level according to the following *doubling-rule*:

$$\begin{aligned} m_i &:= 1 && i = 1, \\ m_i &:= 2^{i-1} + 1 && i > 1. \end{aligned} \tag{24}$$

Implicitly, it is always assumed that $m_0 := 0$. More in general, m_i is basically a function $m_i = m(i) : \mathbb{N}_0 \to \mathbb{N}$ such that $m(0) = 0$ and $m(i) < m(i+1)$. Then, we consider the set of points defined as the zero of Chebyshev polynomials, i.e.,

$$\begin{aligned} y_1^{(i)} &:= 0 && m_i = 1, \\ y_j^{(i)} &:= -\cos\left(\frac{\pi(j-1)}{m_i - 1}\right) && m_i > 1,\ \forall\, j = 1, \ldots, m_i. \end{aligned} \tag{25}$$

With this particular points one obtains a nested sets of points, i.e., $Y^i \subset Y^{i+1}$, for all $i \in \mathbb{N}$.

If the Smolyak interpolation is built using Clenshaw-Curtis points, the following formula computes the cardinality of the resulting sparse grid $\mathcal{Y}(\ell, M)$ for $M > 1$ (see [NTW08b, Eq. (3.24)]).

$$\#\mathcal{Y}(\ell, M) = \sum_{\boldsymbol{i} \in X(\ell, M)} \prod_{m=1}^{M} r(i_m) \quad \text{with} \quad r(i) := \begin{cases} 1 & \text{if } i = 1, \\ 2 & \text{if } i = 2, \\ 2^{i-1} & \text{if } i > 2. \end{cases} \tag{26}$$

See, e.g., [Wal06] for a fast construction of Smolyak quadrature weights using Clenshaw-Curtis collocation points.

Example 3. *Let us compute the weights of a one-dimensional quadrature rule on* $[-1, 1]$ *using Clenshaw-Curtis collocation points.*

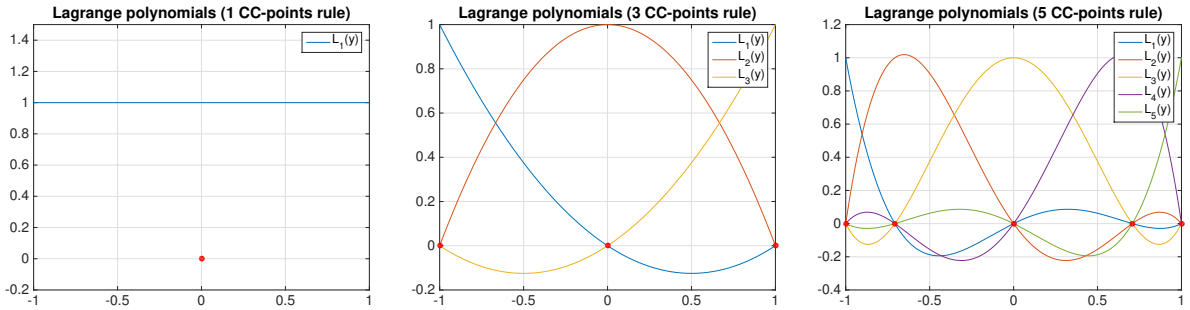


FIGURE 5.1. Lagrange polynomials in $[-1,1]$ at given Clenshaw-Curtis points, 1 (left), 3 (middle), and 5 (right) points, respectively.

- *If* $i=1$, *then* $m_1=1$ *and* $Y^1:=\{y_1^{(1)}\}=\{0\}$, *i.e., there is only* 1 *collocation points. The associated Lagrange polynomial is then the polynomial of degree* 0 *equal to* 1, *i.e.,* $L_1^{(1)}=1$. *Then*

$$w_1^{(1)}=\int_{-1}^{1}L_1^{(1)}(y)dy=\int_{-1}^{1}dy=2.$$

- *If* $i=2$, *then* $m_2=3$ *and* $Y^2:=\{y_1^{(2)},y_2^{(2)},y_3^{(2)}\}=\{-1,0,1\}$. *The associated weights are the integral on* $[-1,1]$ *of the associated* 3 *Lagrange polynomials* $\{L_j^{(2)}\}_{j=1}^3$ *of degree* 2*:*

$$w_1^{(2)}=\int_{-1}^{1}L_1^{(2)}(y)dy=\int_{-1}^{1}\left(\prod_{k=1,k\neq 1}^{3}\frac{y-y_k^{(2)}}{y_1^{(2)}-y_k^{(2)}}\right)dy=\int_{-1}^{1}\left(\frac{1}{2}y^2-\frac{1}{2}y\right)dy=0.\overline{3},$$

$$w_2^{(2)}=\int_{-1}^{1}L_2^{(2)}(y)dy=\int_{-1}^{1}\left(\prod_{k=1,k\neq 2}^{3}\frac{y-y_k^{(2)}}{y_2^{(2)}-y_k^{(2)}}\right)dy=\int_{-1}^{1}\left(-y^2+1\right)dy=1.\overline{3},$$

$$w_3^{(2)}=\int_{-1}^{1}L_3^{(2)}(y)dy=\int_{-1}^{1}\left(\prod_{k=1,k\neq 3}^{3}\frac{y-y_k^{(2)}}{y_3^{(2)}-y_k^{(2)}}\right)dy=\int_{-1}^{1}\left(\frac{1}{2}y^2+\frac{1}{2}y\right)dy=0.\overline{3}.$$

- *If* $i=3$, *then* $m_3=5$ *and* $Y^3:=\{y_1^{(3)},y_2^{(3)},y_3^{(3)},y_4^{(3)},y_5^{(3)}\}=\{-1,-\sqrt{2}/2,0,\sqrt{2}/2,1\}$. *The associated weights are the integral on* $[-1,1]$ *of the associated* 5 *Lagrange polynomials* $\{L_j^{(3)}\}_{j=1}^5$ *of degree* 4*:*

$$w_1^{(3)} = \int_{-1}^{1} L_1^{(3)}(y)dy = \int_{-1}^{1} \left(y^4 - y^3 - \frac{1}{2}y^2 + \frac{1}{2}y\right) dy = 0.0\overline{6},$$

$$w_2^{(3)} = \int_{-1}^{1} L_2^{(3)}(y)dy = \int_{-1}^{1} \left(-2y^4 + \sqrt{2}y^3 + 2y^2 - \sqrt{2}y\right) dy = 0.5\overline{3},$$

$$w_3^{(3)} = \int_{-1}^{1} L_3^{(3)}(y)dy = \int_{-1}^{1} \left(2y^4 - 3y^2 + 1\right) dy = 0.8,$$

$$w_4^{(3)} = \int_{-1}^{1} L_4^{(3)}(y)dy = \int_{-1}^{1} \left(-2y^4 - \sqrt{2}y^3 + 2y^2 + \sqrt{2}y\right) dy = 0.5\overline{3},$$

$$w_5^{(3)} = \int_{-1}^{1} L_5^{(3)}(y)dy = \int_{-1}^{1} \left(y^4 + y^3 - \frac{1}{2}y^2 - \frac{1}{2}y\right) dy = 0.0\overline{6}.$$

Notice that the weights are "symmetric" with respect to the weight associated to the node in the middle, i.e., 0.

6. Illustrative examples

In this section, we explicitly compute the Smolyak interpolator operators (12) and quadrature rule (18) using the the Clenshaw-Curtis sparse grid collocation points for the case of $M = 2$ and levels $\ell = 0, 1, 2$. To this end, we will use the Lagrange polynomials and corresponding weights computed in previous Example 3.

Let $\boldsymbol{i} = (i_1, i_2) \in \mathbb{N}^2$, $\boldsymbol{y} = (y_1, y_2) \in \Gamma^2$ and let $u \in C^0(\Gamma^2; H_0^1(D))$.

Example 4. *Let $\ell = 0$. We have the following sets*

$$X(0,2) := \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ \sum_{n=1}^{2} (i_n - 1) \leq 0 \right\} = \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ |\boldsymbol{i}| \leq 2 \right\} = \{(1,1)\},$$

$$\widetilde{X}(0,2) := \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ \sum_{n=1}^{2} (i_n - 1) = 0 \right\} = \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ |\boldsymbol{i}| = 2 \right\} = \{(1,1)\},$$

$$Y(0,2) := \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ 0 - 2 + 1 \leq \sum_{n=1}^{2} (i_n - 1) \leq 0 \right\} = \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ 1 \leq |\boldsymbol{i}| \leq 2 \right\} = \{(1,1)\}.$$

The Smolyak interpolator operator (12) *is then given by*

$$\mathcal{A}(0,2)(u)(\boldsymbol{y}) = (\Delta^1 \otimes \Delta^1)(u)(\boldsymbol{y}) = \big((\mathcal{U}^1 - \mathcal{U}^0) \otimes (\mathcal{U}^1 - \mathcal{U}^0)\big)(u)(\boldsymbol{y}) = (\mathcal{U}^1 \otimes \mathcal{U}^1)(u)(\boldsymbol{y}).$$

The sparse grid is given by

$$\mathcal{Y}(0,2) = Y^1 \otimes Y^1,$$

where $Y^1 = \{y_1^{(1)}\} = \{0\}$, i.e., there is just one collocation point, $\mathcal{Y}(0,2) = \{(0,0)\}$. Therefore, we have

$$\mathcal{A}(0,2)(u)(\boldsymbol{y}) = \sum_{j_1=1}^{1} \sum_{j_2=1}^{1} u(y_{j_1}^{(1)}, y_{j_2}^{(1)}) L_{j_1}^{(1)}(y_1) L_{j_2}^{(1)}(y_2) = u(y_1^{(1)}, y_1^{(1)}) L_1^{(1)}(y_1) L_1^{(1)}(y_2) = u(0,0),$$

since $L_1^{(1)} = 1$. *The corresponding quadrature rule* (18) *is simply given by integration on both sides of the above equation, i.e.,*

$$\mathcal{Q}(0,2)(u)(\boldsymbol{y}) = \sum_{j_1=1}^{1}\sum_{j_2=1}^{1} w_{j_1}^{(1)} w_{j_2}^{(1)}\, u(y_{j_1}^{(1)}, y_{j_2}^{(1)}) = w_1^{(1)} w_1^{(1)}\, u(y_1^{(1)}, y_1^{(1)}) = 4u(0,0),$$

as $w_1^{(1)} = 2$.

Example 5. *Let* $\ell = 1$. *We have the following sets*

$$X(1,2) := \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ \sum_{n=1}^{2} (i_n - 1) \le 1 \right\} = \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ |\boldsymbol{i}| \le 3 \right\} = \{(1,1),(1,2),(2,1)\},$$

$$\widetilde{X}(1,2) := \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ \sum_{n=1}^{2} (i_n - 1) = 1 \right\} = \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ |\boldsymbol{i}| = 3 \right\} = \{(1,2),(2,1)\},$$

$$Y(1,2) := \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ 1 - 2 + 1 \le \sum_{n=1}^{2} (i_n - 1) \le 1 \right\} = \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ 2 \le |\boldsymbol{i}| \le 3 \right\} = X(1,2).$$

The Smolyak interpolator operator (12) *is then given by*

$$\begin{aligned} \mathcal{A}(1,2)(u)(\boldsymbol{y}) &= \sum_{\boldsymbol{i} \in X(1,2)} (\Delta^{i_1} \otimes \Delta^{i_2})(u)(\boldsymbol{y}) \\ &= \big((\Delta^1 \otimes \Delta^1) + (\Delta^1 \otimes \Delta^2) + (\Delta^2 \otimes \Delta^1) \big)(u)(\boldsymbol{y}) \\ &= \big((\mathcal{U}^1 \otimes \mathcal{U}^2) + (\mathcal{U}^2 \otimes \mathcal{U}^1) - (\mathcal{U}^1 \otimes \mathcal{U}^1) \big)(u)(\boldsymbol{y}). \end{aligned} \tag{27}$$

Recall that if a set of nested (one-dimensional) collocation points is used, as in Clenshaw-Curtis case, the sparse grid is given by (16) *which uses the set* $\widetilde{X}$ *instead of* Y. *Therefore,*

$$\mathcal{Y}(1,2) = \bigcup_{\boldsymbol{i} \in \widetilde{X}(1,2)} (Y^{i_1} \times Y^{i_2}) = (Y^1 \times Y^2) \cup (Y^2 \times Y^1)$$

where $Y^1 = \{y_1^{(1)}\} = \{0\}$ *and* $Y^2 = \{y_1^{(2)}, y_2^{(2)}, y_3^{(2)}\} = \{-1, 0, 1\}$. *Then, the sparse grid is given by the* 5 *points below:*

$$\mathcal{Y}(1,2) = (\{0\} \times \{-1,0,1\}) \cup (\{-1,0,1\} \times \{0\}) = \{(0,0),(0,-1),(0,1),(-1,0),(1,0)\}.$$

We can now explicitly compute $\mathcal{A}(1,2)$. *From Example 4 we already have the third term* $(\mathcal{U}^1 \otimes \mathcal{U}^1)(u)(\boldsymbol{y})$ *in* (27),

$$(\mathcal{U}^1 \otimes \mathcal{U}^1)(u)(\boldsymbol{y}) = u(y_1^{(1)}, y_1^{(1)}) \left[L_1^{(1)}(y_1) L_1^{(1)}(y_2) \right] = u(0,0) \left[L_1^{(1)}(y_1) L_1^{(1)}(y_2) \right].$$

For $(\mathcal{U}^1 \otimes \mathcal{U}^2)$ *we have:*

$$\begin{aligned} (\mathcal{U}^1 \otimes \mathcal{U}^2)(u)(\boldsymbol{y}) &= \sum_{j_1=1}^{1}\sum_{j_2=1}^{3} u(y_{j_1}^{(1)}, y_{j_2}^{(2)}) \left(L_{j_1}^{(1)}(y_1) \otimes L_{j_2}^{(2)}(y_2) \right) = \sum_{j_2=1}^{3} u(0, y_{j_2}^{(2)}) \left(L_1^{(1)}(y_1) \otimes L_{j_2}^{(2)}(y_2) \right) \\ &= u(0,-1) \left[L_1^{(1)}(y_1) L_1^{(2)}(y_2) \right] + u(0,0) \left[L_1^{(1)}(y_1) L_2^{(2)}(y_2) \right] + u(0,1) \left[L_1^{(1)}(y_1) L_3^{(2)}(y_2) \right]. \end{aligned}$$

Analogously, for $(\mathcal{U}^2 \otimes \mathcal{U}^1)$ *we have:*

$$(\mathcal{U}^2 \otimes \mathcal{U}^1)(u)(\boldsymbol{y}) = \sum_{j_1=1}^{3}\sum_{j_2=1}^{1} u(y_{j_1}^{(2)}, y_{j_2}^{(1)}) \left(L_{j_1}^{(2)}(y_1) \otimes L_{j_2}^{(1)}(y_2)\right) = \sum_{j_1=1}^{3} u(y_{j_1}^{(2)}, 0) \left(L_{j_1}^{(2)}(y_1) \otimes L_1^{(1)}(y_2)\right)$$
$$= u(-1,0)\left[L_1^{(2)}(y_1)L_1^{(1)}(y_2)\right] + u(0,0)\left[L_2^{(2)}(y_1)L_1^{(1)}(y_2)\right] + u(1,0)\left[L_3^{(2)}(y_1)L_1^{(1)}(y_2)\right].$$

Putting together the three terms above as in (27)*, we obtain:*

$$\begin{aligned}\mathcal{A}(1,2)(u)(\boldsymbol{y}) &= u(0,0)\left[-L_1^{(1)}(y_1)L_1^{(1)}(y_2) + L_1^{(1)}(y_1)L_2^{(2)}(y_2) + L_1^{(1)}(y_2)L_2^{(2)}(y_1)\right] \\ &= +u(-1,0)\left[L_1^{(2)}(y_1)L_1^{(1)}(y_2)\right] + u(1,0)\left[L_3^{(2)}(y_1)L_1^{(1)}(y_2)\right] \\ &= +u(0,-1)\left[L_1^{(1)}(y_1)L_1^{(2)}(y_2)\right] + u(0,1)\left[L_1^{(1)}(y_1)L_3^{(2)}(y_2)\right] = \sum_{k=1}^{5} u(\boldsymbol{y}_k)\widetilde{L}_k(\boldsymbol{y}),\end{aligned} \tag{28}$$

where $\boldsymbol{y}_k$ *are the sparse collocation points in* $\mathcal{Y}(1,2)$ *and, by recalling the expression for the univariate polynomials* $L_1^{(1)}$ *and* $\{L_k^{(2)}\}$*,* $k = 1,2,3$*, from Example 3,* $\widetilde{L}_k(\boldsymbol{y})$ *are the following basis*

$$\widetilde{L}_1(\boldsymbol{y}) = -1 + (-y_2^2 + 1) + (-y_1^2 + 1),$$
$$\widetilde{L}_2(\boldsymbol{y}) = \frac{1}{2}y_1^2 - \frac{1}{2}y_1, \qquad \widetilde{L}_3(\boldsymbol{y}) = \frac{1}{2}y_1^2 + \frac{1}{2}y_1,$$
$$\widetilde{L}_4(\boldsymbol{y}) = \frac{1}{2}y_2^2 - \frac{1}{2}y_2, \qquad \widetilde{L}_5(\boldsymbol{y}) = \frac{1}{2}y_2^2 + \frac{1}{2}y_2.$$

The associated Smolyak quadrature formula (18) *can be now easily derived by integrating both sides of* (28)*:*

$$\begin{aligned}\mathcal{Q}(1,2)(u)(\boldsymbol{y}) &= u(0,0)\left[-w_1^{(1)}w_1^{(1)} + w_1^{(1)}w_2^{(2)} + w_1^{(1)}w_2^{(2)}\right] \\ &= +u(-1,0)\left[w_1^{(2)}w_1^{(1)}\right] + u(1,0)\left[w_3^{(2)}w_1^{(1)}\right] \\ &= +u(0,-1)\left[w_1^{(1)}w_1^{(2)}\right] + u(0,1)\left[w_1^{(1)}w_3^{(2)}\right] = \sum_{k=1}^{5} u(\boldsymbol{y}_k)\widetilde{w}_k,\end{aligned}$$

where the corresponding weights $\widetilde{w}_k$ *are then given by*

$$\widetilde{w}_1(\boldsymbol{y}) = -(2)^2 + (2)(1.\overline{3}) + (1.\overline{3})(2) = 1.\overline{3},$$
$$\widetilde{w}_2(\boldsymbol{y}) = (0.\overline{3})(2) = 0.\overline{6}, \qquad \widetilde{w}_3(\boldsymbol{y}) = (0.\overline{3})(2) = 0.\overline{6},$$
$$\widetilde{w}_4(\boldsymbol{y}) = (2)(0.\overline{3}) = 0.\overline{6}, \qquad \widetilde{w}_5(\boldsymbol{y}) = (2)(0.\overline{3}) = 0.\overline{6}.$$

Example 6. *Let $\ell = 2$. We have the following sets*

$$X(2,2) := \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ \sum_{n=1}^{2} (i_n - 1) \leq 2 \right\} = \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ |\boldsymbol{i}| \leq 4 \right\}$$

$$= \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\},$$

$$\widetilde{X}(2,2) := \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ \sum_{n=1}^{2} (i_n - 1) = 2 \right\} = \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ |\boldsymbol{i}| = 4 \right\} = \{(1,3),(2,2),(3,1)\},$$

$$Y(2,2) := \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ 2 - 2 + 1 \leq \sum_{n=1}^{2} (i_n - 1) \leq 2 \right\} = \left\{ \boldsymbol{i} \in \mathbb{N}^2 \ : \ 3 \leq |\boldsymbol{i}| \leq 4 \right\}$$

$$= \{(1,2),(1,3),(2,1),(2,2),(3,1)\}.$$

The Smolyak interpolator operator (12) *is then given by*

$$\begin{aligned} \mathcal{A}(2,2) &= \sum_{\boldsymbol{i} \in X(2,2)} (\Delta^{i_1} \otimes \Delta^{i_2}) \\ &= (\Delta^1 \otimes \Delta^1) + (\Delta^1 \otimes \Delta^2) + (\Delta^1 \otimes \Delta^3) + (\Delta^2 \otimes \Delta^1) + (\Delta^2 \otimes \Delta^2) + (\Delta^3 \otimes \Delta^1) \\ &= (\mathcal{U}^1 \otimes \mathcal{U}^3) - (\mathcal{U}^1 \otimes \mathcal{U}^2) + (\mathcal{U}^2 \otimes \mathcal{U}^2) - (\mathcal{U}^2 \otimes \mathcal{U}^1) + (\mathcal{U}^3 \otimes \mathcal{U}^1). \end{aligned} \tag{29}$$

For the sparse grid we have:

$$\mathcal{Y}(2,2) = \bigcup_{\boldsymbol{i} \in \widetilde{X}(2,2)} (Y^{i_1} \times Y^{i_2}) = (Y^1 \times Y^3) \cup (Y^2 \times Y^2) \cup (Y^3 \times Y^1)$$

where $Y^1 = \{y_1^{(1)}\} = \{0\}$, $Y^2 = \{y_1^{(2)}, y_2^{(2)}, y_1^{(2)}\} = \{-1, 0, 1\}$, *and* $Y^3 = \{y_1^{(3)}, y_2^{(3)}, y_3^{(3)}, y_4^{(3)}, y_5^{(3)}\} = \{-1, -\sqrt{2}/2, 0, \sqrt{2}/2, 1\}$. *Therefore the sparse grid consists of the following* 13 *collocation points*

$$\mathcal{Y}(2,2) = \left\{ \begin{matrix} (0,0), & (0,-1), & (1,0), & \left(0, -\frac{\sqrt{2}}{2}\right), & \left(0, \frac{\sqrt{2}}{2}\right), \\ (-1,0), & (1,0), & (-1,-1), & (-1,1), & (1,-1), \\ (1,1), & \left(-\frac{\sqrt{2}}{2}, 0\right), & \left(\frac{\sqrt{2}}{2}, 0\right) & & \end{matrix} \right\}.$$

We can now explicitly compute $\mathcal{A}(2,2)$. *From Example 5 we already have the second and the third terms in* (29). *Let us compute the remaining ones.*

For $(\mathcal{U}^1 \otimes \mathcal{U}^3)$ *we have:*

$$(\mathcal{U}^1 \otimes \mathcal{U}^3)(u)(\boldsymbol{y}) = \sum_{j_1=1}^{1} \sum_{j_2=1}^{5} u(y_{j_1}^{(1)}, y_{j_2}^{(3)}) \left(L_{j_1}^{(1)}(y_1) \otimes L_{j_2}^{(3)}(y_2) \right) = \sum_{j_2=1}^{5} u(0, y_{j_2}^{(3)}) \left(L_1^{(1)}(y_1) \otimes L_{j_2}^{(3)}(y_2) \right)$$

$$\begin{aligned} &= u(0,-1) \left[L_1^{(1)}(y_1) L_1^{(3)}(y_2) \right] + u(0, -\sqrt{2}/2) \left[L_1^{(1)}(y_1) L_2^{(3)}(y_2) \right] + u(0,0) \left[L_1^{(1)}(y_1) L_3^{(3)}(y_2) \right] \\ &+ u(0, \sqrt{2}/2) \left[L_1^{(1)}(y_1) L_4^{(3)}(y_2) \right] + u(0,1) \left[L_1^{(1)}(y_1) L_5^{(3)}(y_2) \right]. \end{aligned}$$

For $(\mathcal{U}^2 \otimes \mathcal{U}^2)$ *we have:*

$$
\begin{aligned}
(\mathcal{U}^2 \otimes \mathcal{U}^2)(u)(\boldsymbol{y}) &= \sum_{j_1=1}^{3}\sum_{j_2=1}^{3} u(y_{j_1}^{(2)}, y_{j_2}^{(2)}) \left(L_{j_1}^{(2)}(y_1) \otimes L_{j_2}^{(2)}(y_2)\right) \\
&= u(-1,-1)\left[L_1^{(2)}(y_1)L_1^{(2)}(y_2)\right] + u(-1,0)\left[L_1^{(2)}(y_1)L_2^{(2)}(y_2)\right] + u(-1,1)\left[L_1^{(2)}(y_1)L_3^{(2)}(y_2)\right] \\
&+ u(0,-1)\left[L_2^{(2)}(y_1)L_1^{(2)}(y_2)\right] + u(0,0)\left[L_2^{(2)}(y_1)L_2^{(2)}(y_2)\right] + u(0,1)\left[L_2^{(2)}(y_1)L_3^{(2)}(y_2)\right] \\
&+ u(1,-1)\left[L_3^{(2)}(y_1)L_1^{(2)}(y_2)\right] + u(1,0)\left[L_3^{(2)}(y_1)L_2^{(2)}(y_2)\right] + u(1,1)\left[L_3^{(2)}(y_1)L_3^{(2)}(y_2)\right].
\end{aligned}
$$

For $(\mathcal{U}^3 \otimes \mathcal{U}^1)$ *we have:*

$$
\begin{aligned}
(\mathcal{U}^3 \otimes \mathcal{U}^1)(u)(\boldsymbol{y}) &= \sum_{j_1=1}^{5}\sum_{j_2=1}^{1} u(y_{j_1}^{(3)}, y_{j_2}^{(1)}) \left(L_{j_1}^{(3)}(y_1) \otimes L_{j_2}^{(1)}(y_2)\right) = \sum_{j_1=1}^{5} u(y_{j_1}^{(3)}, 0) \left(L_{j_1}^{(3)}(y_1) \otimes L_1^{(1)}(y_2)\right) \\
&= u(0,-1)\left[L_1^{(1)}(y_1)L_1^{(3)}(y_2)\right] + u(0,-\sqrt{2}/2)\left[L_1^{(1)}(y_1)L_2^{(3)}(y_2)\right] + u(0,0)\left[L_1^{(1)}(y_1)L_3^{(3)}(y_2)\right] \\
&+ u(0,\sqrt{2}/2)\left[L_1^{(1)}(y_1)L_4^{(3)}(y_2)\right] + u(0,1)\left[L_1^{(1)}(y_1)L_5^{(3)}(y_2)\right].
\end{aligned}
$$

By putting together the above terms according to (29) *we get:*

$$
\begin{aligned}
&\mathcal{A}(2,2)(u)(\boldsymbol{y}) = \\
&u(0,0)\left[L_1^{(1)}(y_1)L_3^{(3)}(y_2) - L_1^{(1)}(y_1)L_2^{(2)}(y_2) + L_2^{(2)}(y_1)L_2^{(2)}(y_2) - L_2^{(2)}(y_1)L_1^{(1)}(y_2) + L_3^{(3)}(y_1)L_1^{(1)}(y_2)\right] \\
&+ u(0,-1)\left[L_1^{(1)}(y_1)L_1^{(3)}(y_2) - L_1^{(1)}(y_1)L_1^{(2)}(y_2) + L_2^{(2)}(y_1)L_1^{(2)}(y_2)\right] \\
&+ u(0,-\sqrt{2}/2)\left[L_1^{(1)}(y_1)L_2^{(3)}(y_2)\right] \;+\; u(0,\sqrt{2}/2)\left[L_1^{(1)}(y_1)L_4^{(3)}(y_2)\right] \\
&+ u(0,1)\left[L_1^{(1)}(y_1)L_5^{(3)}(y_2) - L_1^{(1)}(y_1)L_3^{(2)}(y_2) + L_2^{(2)}(y_1)L_3^{(2)}(y_2)\right] \\
&+ u(-1,0)\left[L_1^{(2)}(y_1)L_2^{(2)}(y_2) - L_1^{(2)}(y_1)L_1^{(1)}(y_2) + L_1^{(3)}(y_1)L_1^{(1)}(y_2)\right] \\
&+ u(-\sqrt{2}/2,0)\left[L_2^{(3)}(y_1)L_1^{(1)}(y_2)\right] \;+\; u(\sqrt{2}/2,0)\left[L_4^{(3)}(y_1)L_1^{(1)}(y_2)\right] \\
&+ u(1,0)\left[L_3^{(2)}(y_1)L_2^{(2)}(y_2) - L_3^{(2)}(y_1)L_1^{(1)}(y_2) + L_5^{(3)}(y_1)L_1^{(1)}(y_2)\right] \\
&+ u(-1,-1)\left[L_1^{(2)}(y_1)L_1^{(2)}(y_2)\right] \;+\; u(-1,1)\left[L_1^{(2)}(y_1)L_3^{(2)}(y_2)\right] \\
&+ u(1,-1)\left[L_3^{(2)}(y_1)L_1^{(1)}(y_2)\right] \;+\; u(1,1)\left[L_3^{(2)}(y_1)L_3^{(2)}(y_2)\right] = \sum_{k=1}^{13} u(\boldsymbol{y}_k)\widetilde{L}_k(\boldsymbol{y}),
\end{aligned}
$$

where the 13 *basis* $\widetilde{L}_k(\boldsymbol{y})$ *can be explicitly recovered using the expression for the univariate Lagrange polynomials from Example 3.*

As in previous examples, the associated Smolyak quadrature formula (18) *is obtained by integrating both sides of the equation above:*

$$\begin{aligned}\mathcal{Q}(2,2)(u)(\boldsymbol{y}) = {} & u(0,0)\left[w_1^{(1)}w_3^{(3)} - w_1^{(1)}w_2^{(2)} + w_2^{(2)}w_2^{(2)} - w_2^{(2)}w_1^{(1)} + w_3^{(3)}w_1^{(1)}\right] \\ & + u(0,-1)\left[w_1^{(1)}w_1^{(3)} - w_1^{(1)}w_1^{(2)} + w_2^{(2)}w_1^{(2)}\right] \\ & + u(0,-\sqrt{2}/2)\left[w_1^{(1)}w_2^{(3)}\right] + u(0,\sqrt{2}/2)\left[w_1^{(1)}w_4^{(3)}\right] \\ & + u(0,1)\left[w_1^{(1)}w_5^{(3)} - w_1^{(1)}w_3^{(2)} + w_2^{(2)}w_3^{(2)}\right] \\ & + u(-1,0)\left[w_1^{(2)}w_2^{(2)} - w_1^{(2)}w_1^{(1)} + w_1^{(3)}w_1^{(1)}\right] \\ & + u(-\sqrt{2}/2,0)\left[w_2^{(3)}w_1^{(1)}\right] + u(\sqrt{2}/2,0)\left[w_4^{(3)}w_1^{(1)}\right] \\ & + u(1,0)\left[w_3^{(2)}w_2^{(2)} - w_3^{(2)}w_1^{(1)} + w_5^{(3)}w_1^{(1)}\right] \\ & + u(-1,-1)\left[w_1^{(2)}w_1^{(2)}\right] + u(-1,1)\left[w_1^{(2)}w_3^{(2)}\right] \\ & + u(1,-1)\left[w_3^{(2)}w_1^{(1)}\right] + u(1,1)\left[w_3^{(2)}w_3^{(2)}\right] = \sum_{k=1}^{13} u(\boldsymbol{y}_k)\widetilde{w}_k(\boldsymbol{y}).\end{aligned}$$

Figure 6.1 shows the 13 *basis* $\widetilde{L}_k$.

7. Stochastic Collocation FEM

For $\Gamma^\infty := [-1,1]^\infty$, consider the parametric model problem

$$\begin{aligned} -\nabla\cdot(a(\boldsymbol{x},\boldsymbol{y})\nabla u(\boldsymbol{x},\boldsymbol{y})) &= f(\boldsymbol{x}) \qquad \boldsymbol{x}\in D,\ \boldsymbol{y}\in\Gamma^\infty, \\ u(\boldsymbol{x},\boldsymbol{y}) &= 0 \qquad \boldsymbol{x}\in\partial D,\ \boldsymbol{y}\in\Gamma^\infty. \end{aligned} \tag{30}$$

with the random field:

$$a(\boldsymbol{x},\boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{m=1}^{\infty} y_m a_m(\boldsymbol{x}), \quad \boldsymbol{y} = (y_m)_{m=1}^{\infty}\in\Gamma^\infty,\ \boldsymbol{x}\in D. \tag{31}$$

The corresponding weak formulation reads as: find $u\in V := L^2_\pi(\Gamma^\infty)\otimes H^1_0(D)$ such that

$$\int_{\Gamma^\infty}\int_D a(\boldsymbol{x},\boldsymbol{y})\nabla u(\boldsymbol{x},\boldsymbol{y})\cdot\nabla v(\boldsymbol{x},\boldsymbol{y})dxd\pi(\boldsymbol{y}) = \int_{\Gamma^\infty}\int_D f(\boldsymbol{x})v(\boldsymbol{x},\boldsymbol{y})dxd\pi(\boldsymbol{y}) \quad \forall\, v\in V. \tag{32}$$

Now, suppose that we truncate the parameter space Γ^∞ for a given positive $M\in\mathbb{N}$, i.e., let $\Gamma^M = [-1,1]^M$. The random field (31) truncated at M is denoted by,

$$a^M(\boldsymbol{x},\boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{m=1}^{M} y_m a_m(\boldsymbol{x}), \quad \boldsymbol{y} = (y_m)_{m=1}^{M}\in\Gamma^M,\ \boldsymbol{x}\in D. \tag{33}$$

The corresponding solution $u^M\in V^M := L^2_\pi(\Gamma^M)\otimes H^1_0(D)$ satisfies the following weak formulation

$$\int_{\Gamma^M}\int_D a^M(\boldsymbol{x},\boldsymbol{y})\nabla u^M(\boldsymbol{x},\boldsymbol{y})\cdot\nabla v(\boldsymbol{x},\boldsymbol{y})dxd\pi(\boldsymbol{y}) = \int_{\Gamma^M}\int_D f(\boldsymbol{x})v(\boldsymbol{x},\boldsymbol{y})dxd\pi(\boldsymbol{y}) \quad \forall\, v\in V^M. \tag{34}$$

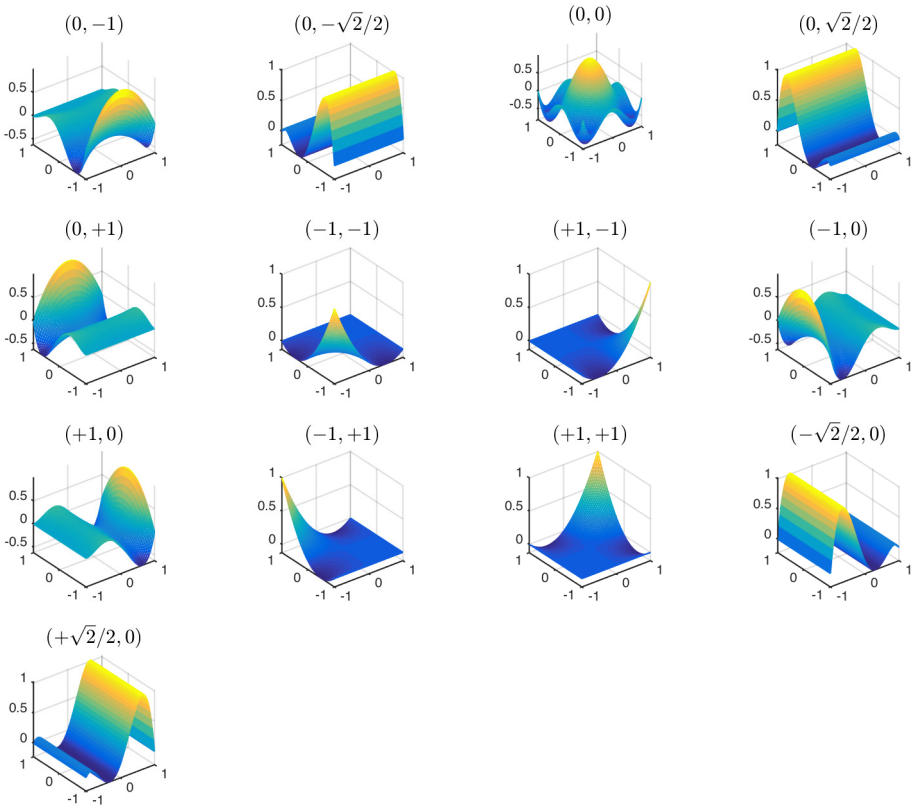


FIGURE 6.1. The set of Lagrange basis function $\{\widetilde{L}_k\}$, $k = 1, \ldots, 13$, of Example 6 for the sparse grid of Clenshaw-Curtis points $\mathcal{Y}(2,2)$. The point above each plot is the collocation point of $\mathcal{Y}(2,2)$ where the corresponding Lagrange basis shown is associated at.

Let $\mathcal{S}_0^1 \subset H_0^1(D)$ be, e.g, a first order (P1) finite element space, and let $u_h^M \in V_h^M := L_\pi^2(\Gamma^M) \otimes \mathcal{S}_0^1$ be corresponding semidiscrete Galerkin approximation satisfying the weak formulation above, i.e.,

$$\int_{\Gamma^M} \int_D a^M(\boldsymbol{x}, \boldsymbol{y}) \nabla u_h^M(\boldsymbol{x}, \boldsymbol{y}) \cdot \nabla v_h(\boldsymbol{x}, \boldsymbol{y}) dx d\pi(\boldsymbol{y}) = \int_{\Gamma^M} \int_D f(\boldsymbol{x}) v_h(\boldsymbol{x}, \boldsymbol{y}) dx d\pi(\boldsymbol{y}) \quad \forall\, v_h \in V_h^M. \tag{35}$$

Differently from the stochastic Galerkin FEM, the *stochastic Collocation Method* selects a suitable set of $Q \in \mathbb{N}$ points $\{\boldsymbol{y}^s\}_{s=1}^Q \in \Gamma^M$, to which is associated a set of finite element Galerkin solutions $u(\boldsymbol{x}, \boldsymbol{y}^s) := u_h^s(\boldsymbol{x}) \in \mathcal{S}_0^1$, for $s = 1, \ldots, Q$, satisfying

$$\int_D a^M(\boldsymbol{x}, \boldsymbol{y}^s) \nabla u_h^s(\boldsymbol{x}) \cdot \nabla v_h(\boldsymbol{x}) dx = \int_D f(\boldsymbol{x}) v_h(\boldsymbol{x}) dx \quad \forall\, v_h \in \mathcal{S}_0^1. \tag{36}$$

Therefore, the "stochastic collocation" fully discrete approximation solution is defined as

$$u_{\mathrm{SC}}(\boldsymbol{x}, \boldsymbol{y}) := \sum_{s=1}^{Q} u_h^s(\boldsymbol{x}) \widetilde{L}_s(\boldsymbol{y}) \ \in \ C^0(\Gamma^M; \mathcal{S}_0^1), \tag{37}$$

where $\{\widetilde{L}_s(\boldsymbol{y})\}_{s=1}^Q$ is the set of the multivariate "Lagrange" polynomials such that $\widetilde{L}_s(\boldsymbol{y}^k) = \delta_{sk}$, for $s, k = 1, \ldots, Q$. In practice, using the same notation of previous sections, the set $\{\boldsymbol{y}^s\}_{s=1}^Q$ is the set of sparse grid collocation points used by the Smolyak construction for some index set

X, and then the solution u_{SC} is the Smolyak interpolant operator $\mathcal{A}_X$ of the solution $u(\boldsymbol{x},\boldsymbol{y}) \in L^2_\pi(\Gamma^M) \otimes H^1_0(D)$ satisfying (34), i.e.,

$$u_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{A}_X(M)(u(\boldsymbol{x},\cdot))(\boldsymbol{y}) = \sum_{\boldsymbol{i}\in X} \Delta^{\boldsymbol{i}}(u(\boldsymbol{x},\cdot))(\boldsymbol{y}). \tag{38}$$

For a more exhaustive overview of the the stochastic collocation method for PDEs with random data see, e.g., [BNT07, Web07, NTW08b, NTW08a, BNTT11, GWZ14, NTTT16, GN17].

7.1. Moments of the stochastic collocation FEM solution. It is easy to compute the moments of the solution u_{SC} in (37) using the Smolyak quadrature rule associated with the set of collocation points.

Let us consider the case of the set of collocation points $\{\boldsymbol{y}^s\}_{s=1}^Q$ associated with a certain level $\ell \in \mathbb{N}_0$, and let $\mathcal{A}(\ell, M)$ be the Smolyak interpolator operator (12). Furthermore, let us focus on the case of mean zero independent uniformly distributed random parameters in Γ^M (i.e., $\boldsymbol{y}^s \sim U(-1,1)$ for all $s = 1,\ldots,Q$). In such case, $d\pi(\boldsymbol{y}) = (1/2)^M d\boldsymbol{y}$ for all $\boldsymbol{y} \in \Gamma^M$.

Mean value. The mean value of u_{SC} is given by:

$$\mathbb{E}\left[u_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})\right] = \mathbb{E}\left[\sum_{s=1}^Q u_h^s(\boldsymbol{x})\widetilde{L}_s(\boldsymbol{y})\right] = \sum_{s=1}^Q u_h^s(\boldsymbol{x}) \int_{\Gamma^M} \widetilde{L}_s(\boldsymbol{y}) d\pi(\boldsymbol{y}).$$

It is clear that the above integral of $\widetilde{L}_s(\boldsymbol{y})$ can be approximated using the associated Smolyak quadrature rule, i.e.,

$$\begin{aligned}
\mathbb{E}\left[\widetilde{L}_s(\boldsymbol{y})\right] = \int_{\Gamma^M} \widetilde{L}_s(\boldsymbol{y}) d\pi(\boldsymbol{y}) &\approx \int_{\Gamma^M} \mathcal{A}(\ell, M)(\widetilde{L}_s)(\boldsymbol{y}) d\pi(\boldsymbol{y}) \\
&= \sum_{r=1}^Q \widetilde{L}_s(\boldsymbol{y}_r) \int_{\Gamma^M} \widetilde{L}_r(\boldsymbol{y}) d\pi(\boldsymbol{y}) \\
&= \left(\frac{1}{2^M}\right) \sum_{r=1}^Q \underbrace{\widetilde{L}_s(\boldsymbol{y}_r)}_{=\delta_{sr}} \underbrace{\int_{\Gamma^M} \widetilde{L}_r(\boldsymbol{y}) d\boldsymbol{y}}_{=\widetilde{w}_r} = \left(\frac{1}{2^M}\right) \widetilde{w}_s.
\end{aligned} \tag{39}$$

Therefore, we get:

$$\mathbb{E}\left[u_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})\right] \approx \left(\frac{1}{2^M}\right) \sum_{s=1}^Q u_h^s(\boldsymbol{x}) \widetilde{w}_s. \tag{40}$$

Variance. The variance of u_{SC} is by definition given by:

$$\mathrm{Var}(u_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})) = \mathbb{E}[u^2_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})] - \mathbb{E}[u_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})]^2. \tag{41}$$

Since the second term can be approximated by (40), let us focus on the first term $\mathbb{E}[u^2_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})]$. We have:

$$\begin{aligned}\mathbb{E}[u^2_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})] &= \int_{\Gamma^M}\left(\sum_{s=1}^{Q} u_h^s(\boldsymbol{x})\widetilde{L}_s(\boldsymbol{y})\right)^2 d\pi(\boldsymbol{y})\\ &= \int_{\Gamma^M}\left(\sum_{s=1}^{Q}(u_h^s(\boldsymbol{x}))^2\widetilde{L}_s^2(\boldsymbol{y}) + 2\sum_{s\neq r} u_h^s(\boldsymbol{x})u_h^r(\boldsymbol{x})\widetilde{L}_s(\boldsymbol{y})\widetilde{L}_r(\boldsymbol{y})\right) d\pi(\boldsymbol{y})\\ &= \sum_{s=1}^{Q}(u_h^s(\boldsymbol{x}))^2\underbrace{\int_{\Gamma^M}\widetilde{L}_s^2(\boldsymbol{y})d\pi(\boldsymbol{y})}_{=:(I)} + 2\sum_{s\neq r} u_h^s(\boldsymbol{x})u_h^r(\boldsymbol{x})\underbrace{\int_{\Gamma^M}\widetilde{L}_s(\boldsymbol{y})\widetilde{L}_r(\boldsymbol{y})d\pi(\boldsymbol{y})}_{=:(II)}\end{aligned} \tag{42}$$

The two integrals (I) and (II) can be approximated again using the Smolyak quadrature rule. Let us consider first (I). We have

$$\begin{aligned}(I) := \mathbb{E}\left[\widetilde{L}_s^2(\boldsymbol{y})\right] &= \int_{\Gamma^M}\widetilde{L}_s^2(\boldsymbol{y})d\pi(\boldsymbol{y})\\ &\approx \int_{\Gamma_M}\mathcal{A}(\ell,M)(\widetilde{L}_s^2)(\boldsymbol{y})d\pi(\boldsymbol{y})\\ &= \left(\frac{1}{2^M}\right)\sum_{r=1}^{Q}\underbrace{\widetilde{L}_s(\boldsymbol{y}_r)\widetilde{L}_s(\boldsymbol{y}_r)}_{=\delta_{sr}^2}\underbrace{\int_{\Gamma^M}\widetilde{L}_r(\boldsymbol{y})d\boldsymbol{y}}_{=\widetilde{w}_r} = \left(\frac{1}{2^M}\right)\widetilde{w}_s = \mathbb{E}[\widetilde{L}_s(\boldsymbol{y})]\end{aligned} \tag{43}$$

Analogously, for the second term (II) we have:

$$\begin{aligned}(II) := \mathbb{E}\left[\widetilde{L}_s(\boldsymbol{y})\widetilde{L}_r(\boldsymbol{y})\right] &= \int_{\Gamma^M}\widetilde{L}_s(\boldsymbol{y})\widetilde{L}_r(\boldsymbol{y})d\pi(\boldsymbol{y})\\ &\approx \int_{\Gamma_M}\mathcal{A}(\ell,M)(\widetilde{L}_s\widetilde{L}_r)(\boldsymbol{y})d\pi(\boldsymbol{y})\\ &= \left(\frac{1}{2^M}\right)\sum_{q=1}^{Q}\underbrace{\widetilde{L}_s(\boldsymbol{y}_q)\widetilde{L}_r(\boldsymbol{y}_q)}_{=\delta_{sq}\delta_{rq}}\int_{\Gamma^M}\widetilde{L}_q(\boldsymbol{y})d\boldsymbol{y} = 0\end{aligned} \tag{44}$$

Putting together the foregoing estimates (42), (43), and (44), the variance (41) can be then approximated by

$$\mathrm{Var}(u_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})) = \left(\frac{1}{2^M}\right)\sum_{s=1}^{Q}(u_h^s(\boldsymbol{x}))^2\widetilde{w}_s \;-\; \mathbb{E}[u_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})]^2. \tag{45}$$

8. Brief considerations about the adaptive stochastic collocation FEM

Our task is to solve problem (34) through the use of stochastic collocation FEM. Usually, the spatial discretisation is always fixed beforehand, i.e., one chooses a fine triangulation $\mathcal{T}$ and construct the finite element space $\mathcal{S}_0^1(\mathcal{T})$. Then, after fixed also $M \in \mathbb{N}$, on the basis of a priori analysis according to which the random field is truncated up to the M "most important" parameters, an adaptive algorithm would produce a set of solutions $\{u_{SC}^{\ell}\}$ for each level $\ell \in \mathbb{N}_0$ of the Smolyak construction. That is, the adaptive just regards the parametric discretization due to the set of collocation points used.

A second choice, which is the one most used so far in literature (at least to our knowledge), constructs the set of solutions $\{u_{SC}^{\ell}\}$ on the basis of Smolyak constructions which depend on adaptively constructed index sets. That is, one firstly fixes an initial index set $X^1 := \{(1,\dots,1)\}$, corresponding to the level $\ell = 0$. Then, instead of using the index sets $X(\ell, M)$, for $\ell = 1, 2, \dots$, which give a predeterminate number of collocation points (see (26) for Clenshaw-Curtis points), some procedure adaptively enriches the current index set X^k, $k \in \mathbb{N}$, inside an adaptive loop. Notice that this approach automatically leads to *anisotropic* algorithms since the adaptively generated index set does not in general consider equally all the M directions.

The second approach just described is the one followed, e.g., by [GG03, CCS14, NTTT16, GN17]. It relies on the concepts of *margin* and *reduced margin* sets. Let us follow [NTTT16] in what follows. Let X be a general index set that should be used by Smolyak interpolation. The *margin* M_X of X is the set of multi-indices $\boldsymbol{i} \in \mathbb{N}^M$ that can be reached with "one step forward" from X, i.e.,

$$M_X := \left\{ \boldsymbol{i} \in \mathbb{N}^M \ : \ \exists\, \boldsymbol{j} \in X \text{ such that } |\boldsymbol{i} - \boldsymbol{j}| = 1 \right\}. \tag{46}$$

The multi-indices of M_X are also called *neighbours* of X. The *reduced margin* R_X of X is the subset of multi-indices of M_X such that "one step backward" in any direction takes into X, i.e.,

$$R_X := \left\{ \boldsymbol{i} \in \mathbb{N}^M \ : \boldsymbol{i} - \boldsymbol{\varepsilon}_j \in X,\ \forall\, j = 1,\dots,M,\ i_j > 1 \right\} \subset M_X. \tag{47}$$

Look at [GN17, Figure 3] for a an example visualisation of these two sets.

The adaptive algorithm should works in a greedy way by assigning during each iteration an error indicator to the indices $\boldsymbol{i}$ of R_X. This error indicator is called *profit* and it is denoted by $P(\boldsymbol{i})$. The analogy with the SGFEM is that the profit of an index $\boldsymbol{i} \in R_X$ can be seen as the single parametric estimate $\|e_{X\mathfrak{Q}}^{(\nu)}\|_{B_0}$ for an index ν in the detail index set $\mathfrak{Q}$. An iteration of the adaptive loop should then perform the following steps:

(*i*) Construct R_X;
(*ii*) Assign a profit $P(\boldsymbol{i})$ to the indices $\boldsymbol{i} \in R_X$;
(*iii*) The index with the highest profit enters into X;
(*iv*) The reduced margin R_X is then updated and the algorithm moves to the next iteration.

Note that Step (*iii*) is what is done, e.g., in [GG03, GN17]. However, in [GN17] the authors themselves suggest the use of some "marking" criterion such as Dörfler strategy.

8.1. Profits of indices. WARNING: THIS PART IS NOT VERY CLEAR

Regarding the form of the profit, in [NTTT16] is done the following.

It is assumed that if the sparse grid interpolation converges π-a.e., i.e., we can write

$$u \equiv \mathcal{A}_X(M)(u)(\boldsymbol{y}) = \sum_{\boldsymbol{i} \in \mathbb{N}^M} \Delta^{\boldsymbol{i}}(u)(\boldsymbol{y}).$$

Then,

$$\mathbb{E}\left[u(\boldsymbol{x},\boldsymbol{y}) - u_{\mathrm{SC}}(\boldsymbol{x},\boldsymbol{y})\right] \overset{(38)}{\leq} \mathbb{E}\left[\sum_{\boldsymbol{i}\notin X} \Delta^{\boldsymbol{i}}(u)(\boldsymbol{y})\right] \leq \sum_{\boldsymbol{i}\notin X} \mathbb{E}\left[\Delta^{\boldsymbol{i}}(u)(\boldsymbol{y})\right] = \sum_{\boldsymbol{i}\notin X} \Delta E(\boldsymbol{i}),$$

where $\Delta E(\boldsymbol{i}) := \mathbb{E}\left[\Delta^{\boldsymbol{i}}(u)\right]$ denotes the *error contribution* of an index $\boldsymbol{i} \in \mathbb{N}^M$ (cf. [NTTT16, Eq. (7)]).

Then, it is introduced the concept of *work contribution* $\Delta W(\boldsymbol{i})$ which is defined as the the number of extra evaluations implied by the addition of the hierarchical surplus $\Delta^{\boldsymbol{i}}(u)$ to the sparse grid approximation for the index set $X \setminus \{\boldsymbol{i}\}$, i.e.,

$$\Delta W(\boldsymbol{i}) := \prod_{k=1}^{M} (m_{i_k} - m_{i_k - 1}), \quad \boldsymbol{i} \in \mathbb{N}^M, \tag{48}$$

(cf. [NTTT16, Eq. (10)]). Here, m_i denote the number of collocation points for one-dimensional rules; for example, for Clenshaw-Curtis points, m_i is defined by (24). Definition (48) holds for sets of nested collocation points. Notice that the cardinality of the sparse grid is equal to the sum of the work contributions of the indices of X, i.e., $Q = \sum_{\boldsymbol{i} \in X} \Delta W(\boldsymbol{i})$.

The profit $P(\boldsymbol{i})$ for the indices $\boldsymbol{i} \in R_X$ is defined as

$$P(\boldsymbol{i}) := \frac{\Delta E(\boldsymbol{i})}{\Delta W(\boldsymbol{i})}. \tag{49}$$

Notice that the error contribution $\Delta E(\boldsymbol{i})$ is not a number but it is a spatial function (see example below); of course, it would be a number if u was a function only dependent on the parameters. There should be some missing norm here.... In fact, notice that the definition of $\Delta E(\boldsymbol{i})$ in, e.g., [GN17], is different (see Eq. (17) inside) as it comes from an upper bound for $\|u - \mathcal{A}_X(u)\|_{L^2_\pi(\Gamma^\infty; H^1_0(D))}$ (see Eq. (14) inside).

Example 7. *Let $M = 2$. Consider the case of Clenshaw-Curtis collocation points and suppose that $X = \{(1,1)\}$. From Example 5, we know that this index set yields only the collocation point $\{(0,0)\}$.*

The margin set M_X and the reduced margin set R_X coincide, $M_X = R_X = \{(1,2),(2,1)\}$. Let us compute the profits $P(\boldsymbol{i})$ for the two indices $\boldsymbol{i} \in R_X$.

Let first $\boldsymbol{i} = (1,2)$. The error contributions is given by

$$\Delta E(\boldsymbol{i}) = \mathbb{E}\left[\Delta^{(1,2)}(u)\right] = \mathbb{E}\left[(\mathcal{U}^1 \otimes \mathcal{U}^2)(u) - (\mathcal{U}^1 \otimes U^1)(u)\right]$$

Again, from Example 5 we have that

$$\begin{aligned}
\mathbb{E}\left[(\mathcal{U}^1 \otimes \mathcal{U}^2)(u)\right] &= u(\boldsymbol{x}, 0, -1)\mathbb{E}\left[L_1^{(1)}(y_1) L_1^{(2)}(y_2)\right] + u(\boldsymbol{x}, 0, 0)\mathbb{E}\left[L_1^{(1)}(y_1) L_2^{(2)}(y_2)\right] \\
&\quad + u(\boldsymbol{x}, 0, 1)\mathbb{E}\left[L_1^{(1)}(y_1) L_3^{(2)}(y_2)\right] \\
&= u(\boldsymbol{x}, 0, -1)\left(\frac{1}{2^M}\right) w_1^{(1)} w_1^{(2)} + u(\boldsymbol{x}, 0, 0)\left(\frac{1}{2^M}\right) w_1^{(1)} w_2^{(2)} \\
&\quad + u(\boldsymbol{x}, 0, 1)\left(\frac{1}{2^M}\right) w_1^{(1)} w_3^{(2)},
\end{aligned}$$

and

$$\mathbb{E}\left[(\mathcal{U}^1 \otimes \mathcal{U}^1)(u)\right] = u(\boldsymbol{x}, 0, 0)\mathbb{E}\left[L_1^{(1)}(y_1) L_1^{(1)}(y_2)\right] = u(\boldsymbol{x}, 0, 0)\left(\frac{1}{2^M}\right)(w_1^{(1)})^2.$$

The work contribution is instead given by

$$\Delta W(\boldsymbol{i}) = (m_1 - m_0)(m_2 - m_1) = (1 - 0)(3 - 1) = 2.$$

Notice that, as already said above, the error contribution $\Delta E(\boldsymbol{i})$ is not a number but a spatial function. In fact, the guys $u(\boldsymbol{x}, 0, 0)$, $u(\boldsymbol{x}, 0, -1)\ldots$, above, are the Galerkin solutions $u_h^{(0,0)}(\boldsymbol{x})$,

$u_h^{(-1,0)}(\boldsymbol{x})$,..., *at the collocation points* $(0,0)$, $(-1,0)$..., *respectively. I don't fully understand how this is used in* [NTTT16].

9. MATLAB FOLDER `STOCHCOL_DIFFUSION`

We report the list of (some of the) functions inside the folder `stochcol_diffusion` of TIFISS. Look at the descriptions inside each functions for more details on how they work.

- `[W] = stochcol_workindex(X,rule)`
  Returns the works (48) of the indices $\boldsymbol{i}$ of the index set X according to the (one-dimensional) quadrature rule considered. Default (and also only implemented) rule is `'cc'` which stands for Clenshaw-Curtis. The input index set has to be a $n \times M$ matrix, where n is the number of indices.
- `[xy,w] = stochcol_spgrid(varargin)`
  Computes the sparse grid collocation points `xy` and the associated weights `w`. Default (and also only implemented) rule is `'cc'` which stands for Clenshaw-Curtis. Type `help stochcol_spgrid` for information about the inputs.
- `[AX] = stochcol_smolyakterms(X)`
  Returns the contributing terms of the Smolyak interpolator operator (12) associated with the input index set `X`. These terms are returned as a matrix containing on each row the sign and the levels i_k associated to the contributing terms of type $\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_M}$.
  For instance, consider $X = \{(1,1),(1,2)(2,1)\}$ and the associated Smolyak operator of level 1 $\mathcal{A}(1,2)$ from Example 4 given by
  $$\mathcal{A}(0,2) = (\mathcal{U}^1 \otimes \mathcal{U}^2) + (\mathcal{U}^2 \otimes \mathcal{U}^1) - (\mathcal{U}^1 \otimes \mathcal{U}^1)$$
  Then, calling `[AX] = stochcol_smolyakterms([1 1; 1 2; 2 1])` returns
  $$\texttt{AX} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$
  This function is called by `stochcol_smolyakterms`.
- `[ncp] = stochcol_sizegrid(w,M,X)`
  Returns the size of the Clenshaw-Curtis sparse grid, i.e., the number of collocation points of the grid of level `w` and dimension `M` using either (24) (for `M` $= 1$) or (26) (for `M` > 1). The computation of X is done inside the function, i.e., if X is not at hand, then calling as `stochcol_sizegrid(w,M)` will firstly computes X and returns the size. If X is at hand, calling `stochcol_sizegrid(w,M,X)` then avoids its computations and simply returns the size.
- `[L,w] = stochcol_polylag_(yy)`
  Returns the univariate Lagrange polynomials `L` (in form of anonymous functions) and the associated weights `w` in $[-1,1]$ for set of univariate Clenshaw-Curtis collocation points `yy`; see Example 3. The weights, i.e., the integrals of Lagrange polynomials, are computed using the faster recursive formulas (2.4)-(2.5) in [Wal06].
  This function is called by `stochcol_lagpoly1D`.
- `stochcol_plotspgrid(xycp,gridlevel)`
  Takes in input the set of collocation points `xycp` and the level `gridlevel` and plots the sparse grid for dimensions $M = 1, 2, 3$.

- `[nodes] = stochcol_nodes_cc(levels)`
  Returns the cell `nodes` of (one-dimensional) Clenshaw-Curtis points for levels `levels` using (24)–(25). Here, `levels` can be a vector, i.e., `[nodes] = stochcol_nodes_cc([1,2,3])`, gives

$$\texttt{nodes\{1\}} = 0$$
$$\texttt{nodes\{2\}} = -1, 0, 1$$
$$\texttt{nodes\{3\}} = -1, -0.7071, 0, 0.7071, 1.$$

- `[MX,RMX] = stochcol_marginindexset(X,iplot)`
  For an input index set `X`, it returns both the margin `MX` and the reduced margin `RMX` index sets given by (46) and (47), respectively. If `iplot` is equal to 1 then it also plot `X`, `MX`, and `RX` if dimension is either 1, 2, or 3.
- `stochcol_lagpoly1D(nlev,rule)`
  Returns the collocation points, weights, and Lagrange polynomials for the rule `rule` up to level `nlev`. Here, default (and also the only implemented) rule is `'cc'` which stands for Clenshaw-Curtis. Data are saved and stored into the corresponding folder `stochcol_diffusion/stochcoldatafiles/cc`.
- `[X] = stochcol_getindexset(w,M,iplot)`
  Returns the index set `X` defined by (9) of level `w` and dimension `M`. If `iplot` is equal to 1 the index set is plotted if $M = 1, 2, 3$.

For an example of stochastic collocation FEM for PDEs with random data run the main driver `stochcol_testproblem_unix`. Problems 5 and 6 solve the model problem on the L-shaped domain using truncated Eigel's random field and truncated Nobile-Tempone-Webster's random field described in [NTW08b], respectively. These random field are implemented into `stochcol_eigel_coeff` and `stochcol_ntw08_coeff`, respectively.

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