$$a(x,y) = a_0(x) + \sum_{m=1}^{\infty} y_m\, a_m(x)$$

$$\mathcal{Y} := \{ \nu = (\nu_1, \nu_2, \dots) \in \mathbb{N}_0^{\mathbb{N}} \mid \#\mathrm{supp}(\nu) < \infty \}, \quad \mathrm{supp}(\nu) = \{ m \in \mathbb{N} \mid \nu_m \neq 0 \}$$

$\mathcal{P} = P \subset \mathcal{Y}$

$$P_\nu(y) = \prod_{m=1}^{\infty} P_{\nu_m}^{m}(y_m) \quad \Rightarrow \quad \boxed{\mathcal{P}_P := \mathrm{span}\{ P_\nu,\ \nu \in P \}}$$

Ex: $P \subset \mathcal{Y}$, $P = \{ \nu^{(1)}, \nu^{(2)}, \nu^{(3)}, \nu^{(4)} \}$ with $\nu^{(1)} = (0,0,\dots)$, $\nu^{(2)} = (1,0,\dots)$, $\nu^{(3)} = (0,1,0,\dots)$, $\nu^{(4)} = (2,0,\dots)$

$\to \mathrm{supp}(\nu^{(1)}) = \emptyset \quad \#\mathrm{supp}(\nu^{(1)}) = 0 < \infty$

$\to \mathrm{supp}(\nu^{(2)}) = \{1\} \Rightarrow \#_ = 1$

$\to \mathrm{supp}(\nu^{(3)}) = \{2\} \Rightarrow \#_ = 1$

$\to \mathrm{supp}(\nu^{(4)}) = \{1,2\} \Rightarrow \#_ = 2$

$$\mathcal{P}_P = \mathrm{span}\{ P_\nu,\ \nu \in P \}$$

$$= \mathrm{span}\{ P_{\nu^{(1)}}, \dots, P_{\nu^{(4)}} \}$$

$$P_{\nu^{(1)}} = \prod_{m \in \mathrm{supp}(\nu^{(1)})} P_{\nu_m}(y_m) = 1$$

$$P_{\nu^{(2)}} = \;\|\; = P_1(y_1)$$

$$P_{\nu^{(3)}} = \;\|\; = P_2(y_2)$$

$$P_{\nu^{(4)}} = \;\|\; = P_2(y_1) P_1(y_2)$$

Let $f(x) = [p(x)]^2$ with $p(x)$ polynom

$f(x) = f(-x)$?

$f(-x) = [p(-x)]^2 \overset{?}{=} [p(x)]^2$

General 3-Term recurrence Formula

$$\boxed{P_j(y) = (a_j y + b_j) P_{j-1}(y) - c_j P_{j-2}(y)} \quad j = 1, 2, \dots$$

$$\equiv a_j y P_{j-1}(y) + b_j P_{j-1}(y) - c_j P_{j-2}(y)$$

~~$P_{j+1}(y) = (a_{j+1} y + b_{j+1}) P_{j-2}(y) + c_{j-1} P_{j-3}(y)$~~

~~$P_j(y) = (a_j y + b_j)\left[(a_{j-1} y + b_{j-1}) P_{j-2}(y) + c_{j-1} P_{j-3}(y) \right] + c_{j-1} P_{j-3}(y)$~~

$$\boxed{c_j = \frac{a_j}{a_{j-1}}} \text{ if } \{P\} \text{ an orthon.}$$

General 3 term-recurrence proprieties

$$\int_\Gamma P_j(y) P_{j-1}(y)\, d\pi(y) = \int_\Gamma (a_j y + b_j) P_{j-1}(y) P_{j-1}(y)\, d\pi(y) - c_j \int_\Gamma P_{j-2} P_{j-1}\, d\pi(y)$$

$P_j \perp P_{j-1}$ (1) $\qquad$ $P_{j-2} \perp P_{j-1}$ (0)

Integrate everything after multipl. by P_{j-1} and get b_j

$$= a_j \int y\, P_{j-1} P_{j-1}\, d\pi(y) + b_j \int P_{j-1} P_{j-1}\, d\pi(y)$$

$\int P_{j-1} P_{j-1} = 1$

$$\Downarrow$$

$$\boxed{b_j = -a_j \langle y P_{j-1}, P_{j-1} \rangle} \quad (*)$$

(3) Integrate after multi. by P_j

$$\int P_j P_j = a_j \int y P_{j-1} P_j + b_j \underbrace{\int P_{j-1} P_j}_{0} - c_j \underbrace{\int P_{j-2} P_j}_{0} = a_j \int y P_{j-1} P_j$$

$$= a_j \int y P_j P_{j-1}$$

$$\Rightarrow \boxed{1 = a_j \langle y P_j, P_{j-1} \rangle} \xrightarrow{\text{Scaling } j \to j-1} \boxed{1 = a_{j-1} \langle y P_{j-1}, P_{j-2} \rangle}$$

Also $A(0) = \frac{L^2}{16}$ and $A(L) = \frac{L^2}{12\sqrt{3}} + \frac{(L-L)^2}{16} = \frac{L^2}{12\sqrt{3}} < \frac{L^2}{16} = A(0)$.
Hence the maximum occurs when $x = 0$ i.e. when just a square is formed.

Integrate everything after multip. by P_{j-2} and get c_j — substitute

(2) $\int P_j P_{j-2} = a_j \int y P_{j-1} P_{j-2} + b_j \underbrace{\int P_{j-1} P_{j-2}}_{0} - c_j \underbrace{\int P_{j-2} P_{j-2}}_{1}$

$$\Rightarrow \quad c_j = a_j \int y P_{j-1} P_{j-2} = a_j \langle y P_{j-1}, P_{j-2} \rangle$$

$$\boxed{c_j = \frac{a_j}{a_{j-1}}} \; *$$

π is even: $[-1, 1]$

$$b_j = -a_j \langle y P_{j-1}, P_{j-1} \rangle = -a_j \int_{-1}^{1} y P_{j-1}(y) P_{j-1}(y) \, d\pi(y)$$

$$= -a_j \int_{-1}^{1} y (P_{j-1}(y))^2 \pi(y) \, dy$$

if even function (as $\pi(y) = \frac{1}{2}$, $\pi(y) = \pi(-y)$) Legendre

Then

$$f(y) = y (P_{j-1}(y))^2 \pi(y)$$

$$f(-y) = (-y) [P_{j-1}(y)]^2 \pi(-y) =$$

$$= -y [P_{j-1}(y)]^2 \pi(y)$$

$$\int_{-1}^{1} y (P_{j-1}(y))^2 \pi(y) dy = \int_{-1}^{0} y (P_{j-1}(y))^2 \pi(y) dy$$

$$+$$

$P(x) = x+1 \to P(-x) = -x+1$

$[P(x)]^2 = (x+1)^2 \to [P(-x)]^2 = (-x+1)^2$

$= x^2+1+2x \qquad = x^2+1-2x$

$y(x) = (P(x))^2$