Problem: find orthonormal poly. $\{P_n\}$ such that

$$\int_{-1}^{1} P_k(x)\, P_m(x)\, \frac{1}{2}\, dx = \delta_{km}$$

i.e. the measure here is $d\pi(x) = \frac{1}{2}\, dx$.

If I choose $\{\sqrt{2}\,\hat{L}_k(x)\}$, i.e. $\sqrt{2}$ times $\{\hat{L}_k\}$ being the Legendre orthonormal pol.

$$\int_{-1}^{1} \left(\sqrt{2}\,\hat{L}_k(x)\right)\left(\sqrt{2}\,\hat{L}_m(x)\right) \frac{1}{2}\, dx = \int_{-1}^{1} 2\, \hat{L}_k(x)\, \hat{L}_m(x)\, \frac{1}{2}\, dx = \int_{-1}^{1} \hat{L}_k\, \hat{L}_m\, dx = \delta_{km}$$

So: let's define $\bar{L}_k(x) \doteq \sqrt{2}\,\hat{L}_k(x)$ $k \geq 0$ $\Rightarrow$ $\int_{-1}^{1} \bar{L}_k(x)\, \bar{L}_m(x)\, \frac{1}{2}\, dx = \delta_{km}$

$$\bar{L}_0(x) = \sqrt{2}\,\hat{L}_0(x) = \sqrt{2}\cdot\frac{1}{\sqrt{2}} L_0(x) = L_0(x) = 1 \qquad L_0$$

$$\bar{L}_1(x) = \sqrt{2}\,\hat{L}_1(x) = \sqrt{2}\cdot\sqrt{\frac{3}{2}}\, L_1(x) = \sqrt{3}\, L_1(x) = \sqrt{3}\, x \qquad L_1$$

$$\vdots$$

$$\bar{L}_{k+1}(x) = \sqrt{2}\,\hat{L}_{k+1}(x) = \frac{(2k+1)\sqrt{2k+3}}{k+1}\, x\, L_k(x) - \frac{k\sqrt{2k+3}}{k+1}\, L_{k-1}(x), \quad \forall k \geq 1 \qquad L_{k+1}$$

Test: $k=1 \Rightarrow$ $\bar{L}_2(x) = \sqrt{2}\,\hat{L}_2(x) = \frac{(2\cdot 1+1)\sqrt{2\cdot 1+3}}{1+1}\, x\, L_1(x) - \frac{1\sqrt{2\cdot 1+3}}{1+1} L_0(x)$

$$= \frac{3\sqrt{5}}{2}\, x\, \underbrace{L_1(x)}_{x} - \frac{\sqrt{5}}{2}\underbrace{L_0(x)}_{1} = \frac{3\sqrt{5}}{2}x^2 - \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2}\left[3x^2 - 1\right]$$

$$\left(\bar{L}_2, \bar{L}_0\right) = \int_{-1}^{1} \frac{\sqrt{5}}{2}(3x^2-1)\cdot 1 \cdot \frac{1}{2}\, dx = \frac{\sqrt{5}}{4}\int_{-1}^{1}(3x^2-1)\, dx$$

$$= \frac{\sqrt{5}}{4}\left[3\left(\frac{x^3}{3}\right)_{-1}^{1} - x\Big|_{-1}^{1}\right] = \frac{\sqrt{5}}{4}\left[3\left(\frac{1}{3}+\frac{1}{3}\right) - 2\right] = \frac{\sqrt{5}}{4}\left[2-2\right] \frac{\sqrt{5}}{2} \neq 0 \text{ fuck!}$$

$\left(\frac{2}{3}\right)$ $\quad \frac{(1)^3}{3} - \frac{(-1)^3}{3} = 2$ $\quad = 0$

$k=1$ $\quad \bar{L}_2(x) = \frac{\sqrt{2\cdot 1+3}\sqrt{2\cdot 1+1}}{1+1}\, x\, \bar{L}_1(x) - \frac{1\sqrt{2\cdot 1+3}}{(1+1)\sqrt{2\cdot 1-1}}\, \bar{L}_0(x) = \frac{\sqrt{5}\sqrt{3}}{2}\, x\sqrt{3} - \frac{\sqrt{5}}{2\cdot 1}\cdot 1$

$$= \frac{3}{2}\sqrt{5}\, x^2 - \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2}\left(3x^2 - 1\right)$$

$k=1$

$$L_2 = L_{1+1}(x) = \frac{3}{2} x L_1(x) - \frac{1}{2} L_0(x) = \frac{3}{2}x^2 - \frac{1}{2} \Rightarrow (L_2, L_2) = \frac{2}{2(2)+1} \int_{-1}^{1} \delta_{22} = \frac{2}{5}$$

$$\hat{L}_2(x) = \sqrt{\frac{2\cdot 2+1}{2}} L_2(x) = \sqrt{\frac{5}{2}} L_2(x) = \sqrt{\frac{5}{2}} \left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$(\hat{L}_2, \hat{L}_2) = \left(\sqrt{\frac{5}{2}} L_2(x), \sqrt{\frac{5}{2}} L_2(x)\right) = \frac{5}{2} (L_2, L_2) = \frac{5}{2} \cdot \frac{2}{5} = 1 \quad \text{right.}$$

$-2-$

$k \geq 1 \Rightarrow$

$$\boxed{\hat{L}_{k+1}(x) = \sqrt{\frac{2k+3}{2}} L_{k+1}(x)}$$

(1)

$$\hat{L}_0(x) = \frac{1}{\sqrt{2}} L_0(x)$$

(2)

$$\hat{L}_1(x) =$$

$$\boxed{\hat{L}_k(x) = \sqrt{\frac{2k+1}{2}} L_k(x)}$$

$$\hat{L}_0(x) = \sqrt{\frac{2\cdot 0+1}{2}} L_0(x) = \frac{1}{\sqrt{2}} L_0(x)$$

$$\hat{L}_1(x) = \sqrt{\frac{2\cdot 1+1}{2}} L_1(x) = \sqrt{\frac{3}{2}} L_1(x)$$

$$\vdots$$

(3)

$$\hat{L}_{k+1}(x) = \sqrt{\frac{2(k+1)+1}{2}} L_{k+1}(x) = \sqrt{\frac{2k+3}{2}} L_{k+1}(x)$$

$$= \frac{(2k+1)\sqrt{2k+3}}{(k+1)\sqrt{2}} x L_k(x) - \frac{k\sqrt{2k+3}}{(k+1)\sqrt{2}} L_{k-1}(x), \quad k \geq 1$$

(*) $\{\hat{L}_k\}$ is the set of orthonormal Legendre poly. in $(-1,1)$, i.e.

$$\Rightarrow (\hat{L}_k, \hat{L}_m) \equiv \int_{-1}^{1} \hat{L}_k(x) \hat{L}_m(x)\, dx = \delta_{km}$$

$k=1$

$$\hat{L}_2(x) = \frac{(2\cdot 1+1)\sqrt{2\cdot 1+3}}{(1+1)\sqrt{2}} x L_1(x) - \frac{1\sqrt{2\cdot 1+3}}{(1+1)\sqrt{2}} L_0(x) = \frac{3\sqrt{5}}{2\sqrt{2}} x^2 - \frac{\sqrt{5}}{2\sqrt{2}}$$

$$= \sqrt{\frac{5}{2}} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) = \sqrt{\frac{5}{2}} L_2(x) = \sqrt{\frac{2(1+1)+1}{2}} L_2(x)$$