An efficient adaptive algorithm for elliptic problems with random input data

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joint work with Alex Bespalov



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Outline

1. Parametric PDEs and stochastic Galerkin FEM
2. *Hierarchical* a posteriori error estimation
3. Adaptive algorithm
4. Some numerical examples

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Parametric model problem: setup

Find the solution $u : D \times \Gamma \to \mathbb{R}$ to the following *parametric* problem

$$\begin{aligned} -\nabla \cdot (a(x, \mathbf{y}) \nabla u(x, \mathbf{y})) &= f(x) && x \in D \subset \mathbb{R}^2, \quad \mathbf{y} \in \Gamma, \\ u(x, \mathbf{y}) &= 0 && x \in \partial D, \quad \mathbf{y} \in \Gamma, \end{aligned}$$

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- Γ is a *parameter* domain given by $\Gamma := \bigotimes_{m=1}^{\infty} \Gamma_m$, with $\Gamma_m := [-1, 1]$ for all m;
- $\mathbf{y} = (y_1, y_2, \dots) \in \Gamma$, is a multivariate random parameter s.t. $y_m \in \Gamma_m$ for all m;
- the **random field** $a(x, \mathbf{y})$ is represented as (*Karhunen-Loève expansion* type)

$$a(x, \mathbf{y}) = a_0(x) + \sum_{m=1}^{\infty} a_m(x)\, y_m, \qquad x \in D,\ \mathbf{y} \in \Gamma,$$

where $\{a_m\}_{m=0}^{\infty} \in L^\infty(D)$ and $0 < a_{\min} \leq a(x, \mathbf{y}) \leq a_{\max}$ a.e. in $D \times \Gamma$.

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Modelling the uncertainty:

- y_m are **iid** with density function $\rho_m : \Gamma_m \to \mathbb{R}$ (e.g., uniform);
- $\rho(\mathbf{y}) = \rho_1(y_1) \times \rho_2(y_2) \times \cdots : \Gamma \to \mathbb{R}$ is the **joint density function**.

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Weak form: we use the **tensor product** space $V = H_0^1(D) \otimes L_\rho^2(\Gamma) \simeq L_\rho^2(\Gamma; H_0^1(D))$.
Find $u(x, \mathbf{y}) \in V$ s.t.

$$B(u, v) = F(v) \qquad \forall\, v \in V,$$

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where

$$B_0(u, v) := \int_\Gamma \int_D a_0(x) \nabla u(x, \mathbf{y}) \cdot \nabla v(x, \mathbf{y})\, dx\, \rho(\mathbf{y}) d\mathbf{y},$$

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- Assumptions on $a(x, \mathbf{y})$ prove **existence and uniqueness** of the weak solution u;
- B and B_0 are inner products which induce equivalent norms

$$B_0(\cdot, \cdot)^{1/2} =: \|\cdot\|_0 \simeq |||\cdot||| := B(\cdot, \cdot)^{1/2}.$$

Parametric model problem: discrete weak form

We define two spaces $X \subset H_0^1(D)$ and $\mathcal{P}_{\mathcal{P}} \subset L_\rho^2(\Gamma)$ and define the **tensor product** approximation space

$$V_{X\mathcal{P}} := X \otimes \mathcal{P}_{\mathcal{P}} \quad \subset \quad H_0^1(D) \otimes L_\rho^2(\Gamma) = V$$

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$$\mathcal{P} := \Big\{\nu = (\nu_1, \nu_2, \dots) \in \mathbb{N}_0^{\mathbb{N}} \ : \ \#\operatorname{supp}(\nu) < \infty\Big\}.$$

  In particular,

$$P_\nu(\mathbf{y}) := \prod_{m=1}^{\infty} p_{\nu_m}(y_m) = p_{\nu_1}(y_1)p_{\nu_2}(y_2)p_{\nu_3}(y_3)\cdots \qquad \forall\, \nu \in \mathcal{P},$$

  where $\{p_k\}_{k\in 0}$ (with $p_0 := 1$) is an **orthonormal basis** of $L^2_{\rho_m}(\Gamma_m)$ for all m.

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Example: suppose $\mathcal{P}$ contains $\nu := (1, 0, 0, 3, 0, 7, 0, 0, \dots)$, then

$$P_\nu(\mathbf{y}) = \prod_{m=1}^{\infty} p_{\nu_m}(y_m) = p_1(y_1)p_3(y_4)p_7(y_6)$$

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The **discrete weak form** reads as:

$$\text{find } u_{X\mathcal{P}} = \sum_{\nu\in\mathcal{P}} \sum_{i=1}^{n_X} u_{\nu i}\varphi_i P_\nu \in V_{X\mathcal{P}} \quad : \quad B(u_{X\mathcal{P}}, v_{X\mathcal{P}}) = F(v_{X\mathcal{P}}) \qquad \forall\, v_{X\mathcal{P}} \in V_{X\mathcal{P}}.$$

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Idea: use a more accurate SGFEM solution $\widehat{u}_{X\mathcal{P}} \in \widehat{V}_{X\mathcal{P}} \supset V_{X\mathcal{P}}$ to estimate the error:

$$|||\, e \,||| := |||\, u - u_{X\mathcal{P}} \,||| \quad \approx \quad |||\, \widehat{u}_{X\mathcal{P}} - u_{X\mathcal{P}} \,||| =: |||\, \widehat{e}_{X\mathcal{P}} \,|||$$

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Smarter idea: avoid the computation of $\widehat{u}_{X\mathcal{P}}$ by defining $\widehat{V}_{X\mathcal{P}}$ as follows:

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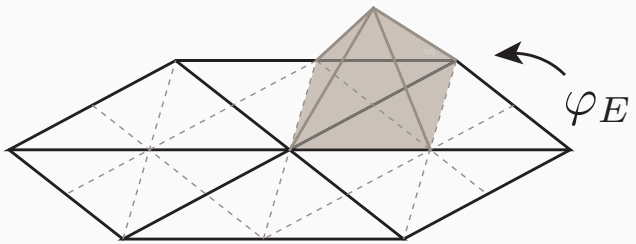
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Typical choice for $Y \subset H^1_0(D)$:

- $Y := \operatorname{span}\{\varphi_E(x) \,:\, E \in \mathcal{E}_{\text{int}}\}$ is a first-order finite element space associated with **edge midpoints** of the mesh.



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Typical choice for $\mathcal{P}_{\mathcal{Q}} \subset L^2_\rho(\Gamma)$ is:

- $\mathcal{P}_{\mathcal{Q}} := \operatorname{span}\{P_\mu(\mathbf{y}) \,:\, \mu \in \mathcal{Q}\}$ where $\mathcal{Q}$ is the "boundary of $\mathcal{P}$" which activates 1 more parameter and increases the polynomial degree by 1, e.g.,

$$\mathcal{P} = \left\{ \begin{matrix} (0,0,0,0\ldots) \\ (\mathbf{1},0,0,0\ldots) \end{matrix} \right\} \qquad \Longrightarrow \qquad \mathcal{Q} = \left\{ \begin{matrix} (0,\mathbf{1},0,0\ldots) \\ (\mathbf{1},\mathbf{1},0,0\ldots) \\ (\mathbf{2},0,0,0\ldots) \end{matrix} \right\}.$$

Hierarchical enrichments: estimates and error reductions

The error $\widehat{e}_{X\mathcal{P}} \in \widehat{V}_{X\mathcal{P}}$ satisfies the residual problem:

$$B(\widehat{e}_{X\mathcal{P}}, \widehat{v}_{X\mathcal{P}}) = F(\widehat{v}_{X\mathcal{P}}) - B(u_{X\mathcal{P}}, \widehat{v}_{X\mathcal{P}}) \qquad \widehat{v}_{X\mathcal{P}} \in \widehat{V}_{X\mathcal{P}}$$

Due to the hierarchical decomposition of $\widehat{V}_{X\mathcal{P}}$ we get two independent residual problems:

$$\begin{aligned}
&\text{find } e_{Y\mathcal{P}} \in V_{Y\mathcal{P}} \quad : \quad B_0(e_{Y\mathcal{P}}, v_{Y\mathcal{P}}) = F(v_{X\mathcal{P}}) - B(u_{X\mathcal{P}}, v_{Y\mathcal{P}}) \qquad \forall\, v_{Y\mathcal{P}} \in V_{Y\mathcal{P}} \\
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\end{aligned}$$

and we define the following error estimate ([BS16][1]):

$$\eta^2_{X\mathcal{P}} := \|e_{Y\mathcal{P}}\|_0^2 + \|e_{X\mathcal{Q}}\|_0^2.$$

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$$\eta^2_{X\mathcal{P}} := \|e_{Y\mathcal{P}}\|^2_0 + \|e_{X\mathcal{Q}}\|^2_0.$$

- $\eta_{X\mathcal{P}}$ is *reliable* and *efficient*, i.e., $\exists\ C_1, C_2 > 0$ s.t. $C_1\, \eta_{X\mathcal{P}} \leq |||\, e\, ||| \leq C_2\, \eta_{X\mathcal{P}}$;

[1] [BS16] **Bespalov, Silvester**, SIAM J. Sci. Comput., 38 (2016), pp. A2118–A2140;

Hierarchical enrichments: estimates and error reductions

The error $\widehat{e}_{X\mathcal{P}} \in \widehat{V}_{X\mathcal{P}}$ satisfies the residual problem:

$$B(\widehat{e}_{X\mathcal{P}}, \widehat{v}_{X\mathcal{P}}) = F(\widehat{v}_{X\mathcal{P}}) - B(u_{X\mathcal{P}}, \widehat{v}_{X\mathcal{P}}) \qquad \widehat{v}_{X\mathcal{P}} \in \widehat{V}_{X\mathcal{P}}$$

Due to the hierarchical decomposition of $\widehat{V}_{X\mathcal{P}}$ we get two independent residual problems:

$$\begin{aligned}
&\text{find } e_{Y\mathcal{P}} \in V_{Y\mathcal{P}} \quad : \quad B_0(e_{Y\mathcal{P}}, v_{Y\mathcal{P}}) = F(v_{X\mathcal{P}}) - B(u_{X\mathcal{P}}, v_{Y\mathcal{P}}) \qquad \forall\, v_{Y\mathcal{P}} \in V_{Y\mathcal{P}} \\
&\text{find } e_{X\mathcal{Q}} \in V_{X\mathcal{Q}} \quad : \quad B_0(e_{X\mathcal{Q}}, v_{X\mathcal{Q}}) = F(v_{X\mathcal{Q}}) - B(u_{X\mathcal{P}}, v_{X\mathcal{Q}}) \qquad \forall\, v_{X\mathcal{Q}} \in V_{X\mathcal{Q}}
\end{aligned}$$

and we define the following error estimate ([BS16][1]):

$$\eta^2_{X\mathcal{P}} := \|e_{Y\mathcal{P}}\|_0^2 + \|e_{X\mathcal{Q}}\|_0^2.$$

- $\eta_{X\mathcal{P}}$ is *reliable* and *efficient*, i.e., $\exists\, C_1, C_2 > 0$ s.t. $C_1\, \eta_{X\mathcal{P}} \leq |||\, e\, ||| \leq C_2\, \eta_{X\mathcal{P}}$;
- $\|e_{Y\mathcal{P}}\|_0$ and $\|e_{X\mathcal{Q}}\|_0$ **estimate the error reductions** achieved

$$\|e_{Y\mathcal{P}}\|_0 \;\simeq\; |||\, u_{\widehat{X}\mathcal{P}} - u_{X\mathcal{P}}\, |||, \qquad\qquad \|e_{X\mathcal{Q}}\|_0 \;\simeq\; |||\, u_{X\widehat{\mathcal{P}}} - u_{X\mathcal{P}}\, |||,$$

  if either
  - $u_{\widehat{X}\mathcal{P}} \in V_{X\mathcal{P}} \oplus V_{Y\mathcal{P}}$ (spatial refinement only) or
  - $u_{X\widehat{\mathcal{P}}} \in V_{X\mathcal{P}} \oplus V_{X\mathcal{Q}}$ (parametric enrichment only)

  is computed.

[1] [BS16] **Bespalov, Silvester**, SIAM J. Sci. Comput., 38 (2016), pp. A2118–A2140;

Outline

1 Parametric PDEs and stochastic Galerkin FEM

2 *Hierarchical* a posteriori error estimation

3 Adaptive algorithm

4 Some numerical examples

An "error-reduction" based adaptive algorithm

A typical (stochastic) **adaptive Galerkin FEM algorithm** consist of the following loop:

FOR $\ell = 0, 1, 2, \dots$ DO:

- **SOLVE:** compute the SGFEM approximation $u_\ell \in V_\ell$;
- **ESTIMATE**: compute local *spatial* and *parametric* estimates, as well as the global estimate η_ℓ such that $\eta_\ell \simeq |||\, u - u_\ell \,|||$;
- **MARK:** determine a set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ and marked indices $\mathcal{M}_\ell \subseteq \mathcal{Q}_\ell$;
- **REFINE:** compute the enhanced space $V_{\ell+1}$: refine $\mathcal{M}_\ell$ and add $\mathcal{M}_\ell$.

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SOLVE: solve the (discrete) weak problem:

$$\text{find } u_\ell \in V_\ell \qquad B(u_\ell, v_\ell) = F(v_\ell) \qquad \forall\, v_\ell \in V_\ell = X(\mathcal{T}_\ell) \otimes \mathcal{P}_{\mathcal{P}_\ell}$$

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ESTIMATE: compute *local* spatial and parametric error estimates

- contribution from elements: $\left\{ e_{Y\mathcal{P}}^{(K)}, \quad K \in \mathcal{T}_\ell \right\}$,
- contribution from indices: $\left\{ e_{X\mathcal{Q}}^{(\mu)}, \ \mu \in \mathcal{Q}_\ell \right\}$,

and compute the global estimate:

$$\eta_\ell^2 = \eta(\mathcal{T}_\ell)^2 \ + \ \eta(\mathcal{Q}_\ell)^2 \ := \sum_{K \in \mathcal{T}_\ell} \| e_{Y\mathcal{P}}^{(K)} \|_0^2 \ + \ \sum_{\mu \in \mathcal{Q}_\ell} \| e_{X\mathcal{Q}}^{(\mu)} \|_0^2 .$$

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If $\eta_\ell \leq$ TOL then END ALGORITHM;

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MARK (Version 1): given $\theta_{\mathcal{X}}, \theta_{\mathcal{P}} \in (0, 1]$,

IF $\eta(\mathcal{T}_\ell) \geq \eta(\mathcal{Q}_\ell)$

- define $\mathcal{M}_\ell$ as the minimal set s.t. $\sum_{K \in \mathcal{M}_\ell} \| e_{\mathcal{YP}}^{(K)} \|_0^2 \geq \theta_{\mathcal{X}}\, \eta(\mathcal{T}_\ell)^2$;
- define $\mathcal{M}_\ell := \emptyset$;

  # we want to do a spatial refinement only!

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ELSE

- define $\mathcal{M}_\ell := \emptyset$;
- define $\mathcal{M}_\ell$ as the minimal set s.t. $\sum_{\mu \in \mathcal{M}_\ell} \| e^{(\mu)}_{\mathcal{X}\mathcal{Q}} \|_0^2 \geq \theta_{\mathcal{X}}\, \eta(\mathcal{Q}_\ell)^2$;

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MARK (Version 2): given $\theta_{\mathcal{X}}, \theta_{\mathcal{P}} \in (0, 1]$,

define $\widetilde{\mathcal{M}}_\ell$ as the minimal set s.t. $\sum_{K \in \widetilde{\mathcal{M}}_\ell} \| e^{(K)}_{Y\mathcal{P}} \|_0^2 \geq \theta_{\mathcal{X}}\, \eta(\mathcal{T}_\ell)^2$;

define $\widetilde{\mathcal{M}}_\ell$ as the minimal set s.t. $\sum_{\mu \in \widetilde{\mathcal{M}}_\ell} \| e^{(\mu)}_{X\mathcal{Q}} \|_0^2 \geq \theta_{\mathcal{P}}\, \eta(\mathcal{Q}_\ell)^2$;

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IF $\eta(\widetilde{\mathcal{M}}_\ell) \geq \eta(\widetilde{\mathcal{M}}_\ell)$

- define $\mathcal{M}_\ell := \widetilde{\mathcal{M}}_\ell$ and $\mathcal{M}_\ell := \emptyset$;

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IF $\eta(\widetilde{\mathcal{M}}_\ell) \geq \eta(\widetilde{\mathcal{M}}_\ell)$

- define $\mathcal{M}_\ell := \widetilde{\mathcal{M}}_\ell$ and $\mathcal{M}_\ell := \emptyset$;

  # we want to do a spatial refinement only!

ELSE

- define $\mathcal{M}_\ell := \emptyset$ and $\mathcal{M}_\ell := \widetilde{\mathcal{M}}_\ell$;

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REFINE: for both Version 1 and Version 2 of the MARK step, define

- $\mathcal{T}_{\ell+1} := \mathrm{REFINE}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ (**new triangulation**),
- $\mathcal{P}_{\ell+1} := \mathcal{P}_\ell \cup \mathcal{M}_\ell$ (**new index set**),

then we have the new approximation space

$$V_{\ell+1} = X(\mathcal{T}_{\ell+1}) \otimes \mathcal{P}_{\mathcal{P}_{\ell+1}}.$$

Outline

1. Parametric PDEs and stochastic Galerkin FEM
2. *Hierarchical* a posteriori error estimation
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Example 1 (square domain)

Let $D = (0,1)^2$. Consider

$$\begin{aligned} -\nabla \cdot (\, a(x,\mathbf{y}) \nabla u(x,\mathbf{y}) \,) &= 1 \qquad && x = (x_1, x_2) \in D, \quad \mathbf{y} \in \Gamma, \\ u(x,\mathbf{y}) &= 0 && x = (x_1, x_2) \in \partial D, \ \mathbf{y} \in \Gamma, \end{aligned}$$

with $a(x,\mathbf{y})$ as given in [EGSZ14][1]:

$$a(x,\mathbf{y}) = 1 + \sum_{m=1}^{\infty} y_m \left(\frac{\bar{\alpha}}{m^2} \cos(2\pi\beta_1(m)x_1) \cos(2\pi\beta_2(m)x_2) \right), \qquad y_m \sim U(-1,1) \text{ iid},$$

where $\bar{\alpha} = 0.547$ and

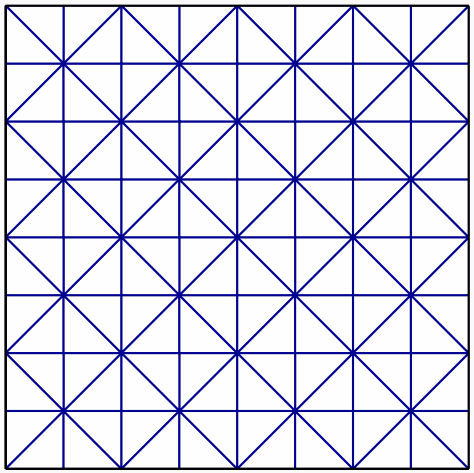
$$\beta_1(m) := m - k(m)(k(m)+1)/2, \quad \beta_2(m) := k(m) - \beta_1(m), \quad k(m) := \left\lfloor -1/2 + \sqrt{1/4 + 2m} \right\rfloor.$$

Initial index set: $\mathcal{P}_0 := \{(0,0,0,\dots), (\mathbf{1},0,0,\dots)\}$;

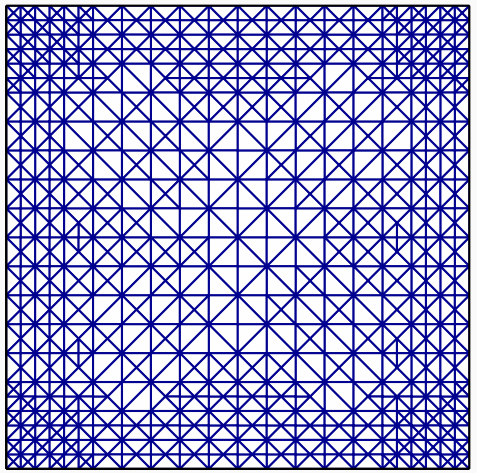
[1] **Eigel, Gittelson, Schwab, Zander**, Comput. Methods. Appl. Mech. Engrg., 270:247–269, 2014

Example 1 (square domain)

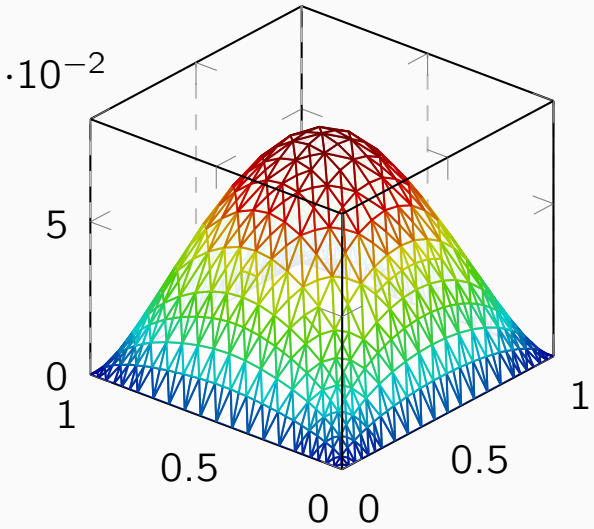
$\#\mathcal{T}_0 = 128$



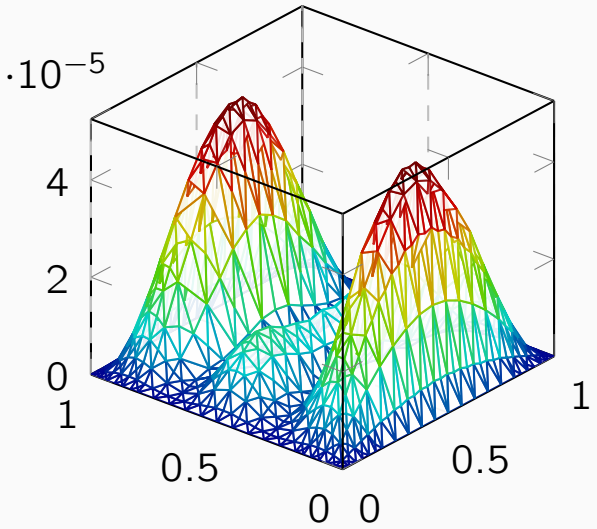
$\#\mathcal{T}_{22} = 1350$



$\mathbb{E}[u_{X\mathcal{P}}]$



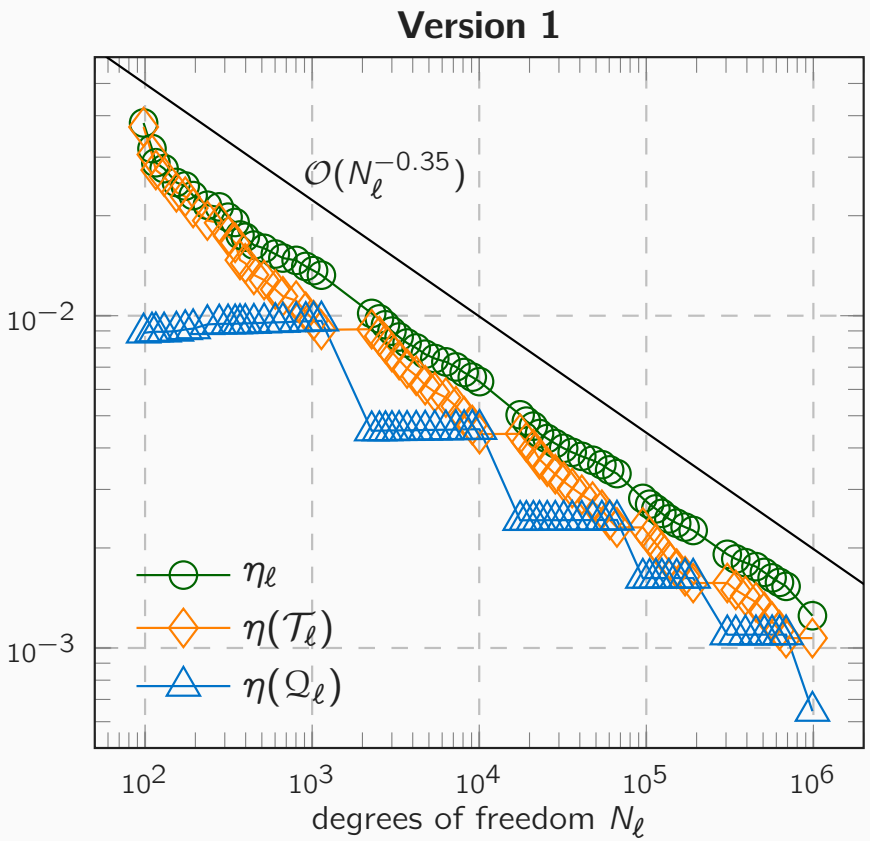
$\mathrm{Var}(u_{X\mathcal{P}})$

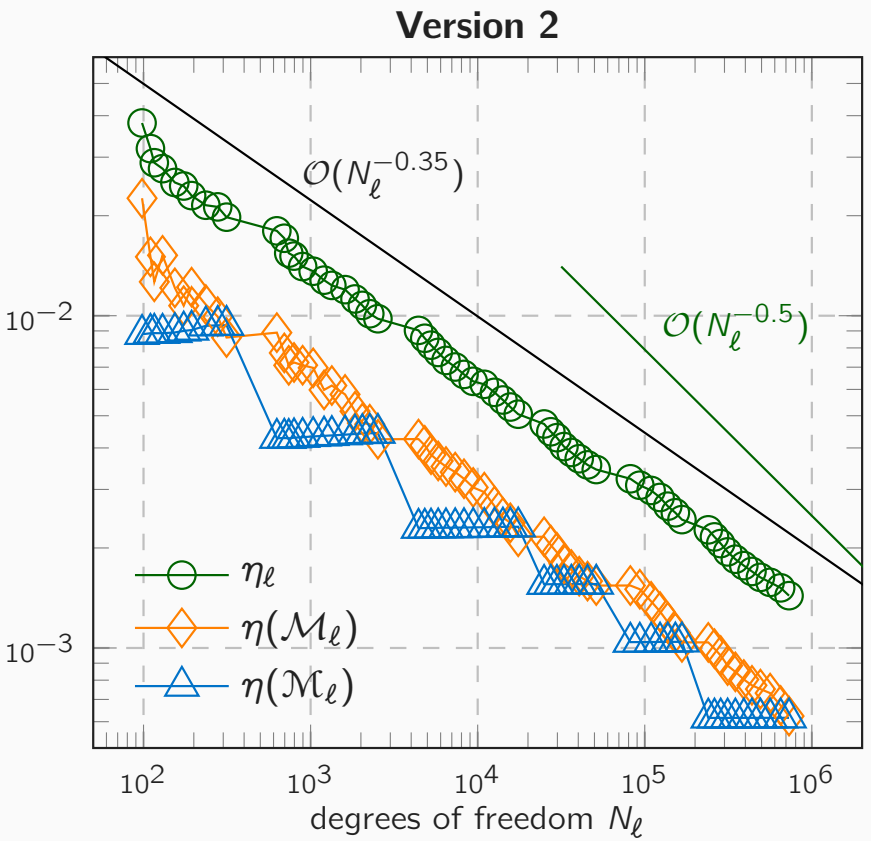


Example 1 (square domain)

- $\theta_{\mathcal{X}} = 0.2$, $\theta_{\mathcal{P}} = 0.9$, and $\mathrm{TOL} = 1.5 \cdot 10^{-3}$;

	time (sec)	#iterations	#elements	#indices	#dofs
Version 1	605	64	86,552	23	986,769
Version 2	**479**	**61**	**64,156**	23	**730,319**



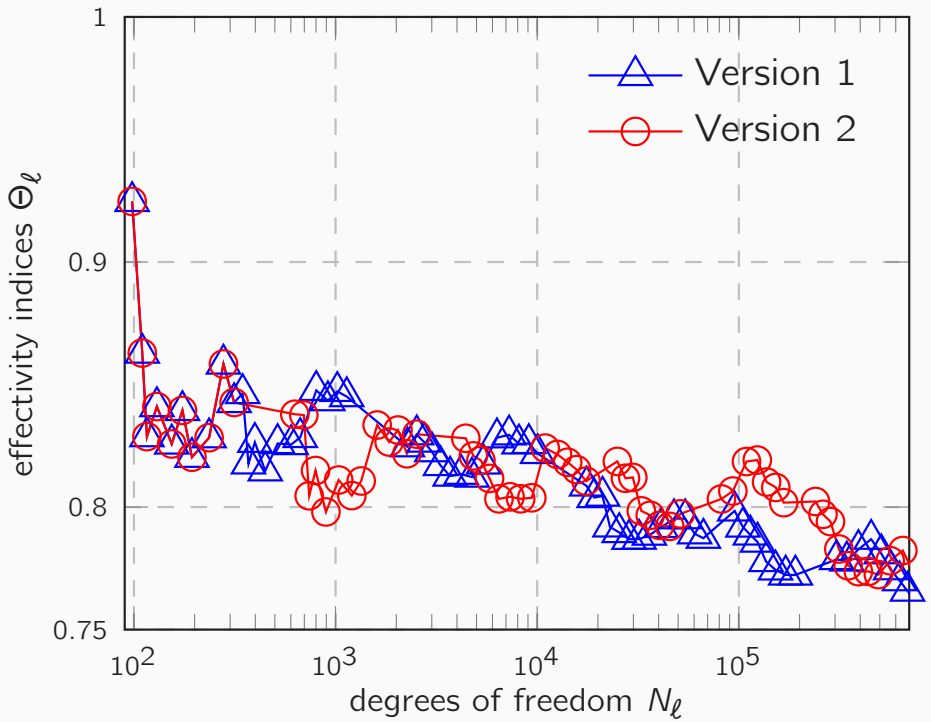


Example 1 (square domain)

Version 1		Version 2	
$\ell = 0$	$\begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}$	$\ell = 0$
$\ell = 21$	$\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{2} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{2} & 0 \end{pmatrix}$	$\ell = 10$
$\ell = 35$	$\begin{pmatrix} 0 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{3} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{3} & 0 & 0 \end{pmatrix}$	$\ell = 23$
$\ell = 48$	$\begin{pmatrix} 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 \\ \mathbf{2} & \mathbf{1} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 \\ \mathbf{2} & \mathbf{1} & 0 & 0 \end{pmatrix}$	$\ell = 36$
$\ell = 56$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{2} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{2} & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{4} & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{2} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{2} & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{4} & 0 & 0 & 0 & 0 \end{pmatrix}$	$\ell = 44$
$\ell = 64$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{2} & 0 & 0 & 0 & 0 \\ \mathbf{2} & 0 & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{3} & 0 & \mathbf{1} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{2} & 0 & 0 & 0 & 0 \\ \mathbf{2} & 0 & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{3} & 0 & \mathbf{1} & 0 & 0 & 0 \end{pmatrix}$	$\ell = 51$
#parameters $= 6$ degree $= 4$		#parameters $= 6$ degree $= 4$	

Efficiency of the error-estimation: reference solution u_{ref} computed with P_2 approximations over a fine mesh and a large index set ($\sim 24 \cdot 10^6$ dofs):

$$\Theta_\ell := \frac{\eta_\ell}{|||\, u_{\text{ref}} - u_\ell \,|||} = \frac{\eta_\ell}{(|||\, u_{\text{ref}} \,|||^2 - |||\, u_\ell \,|||^2)^{1/2}}$$



Example 2 (L-shaped domain)

Let $D = (-1,1)^2 \setminus (-1,0]^2$. Consider

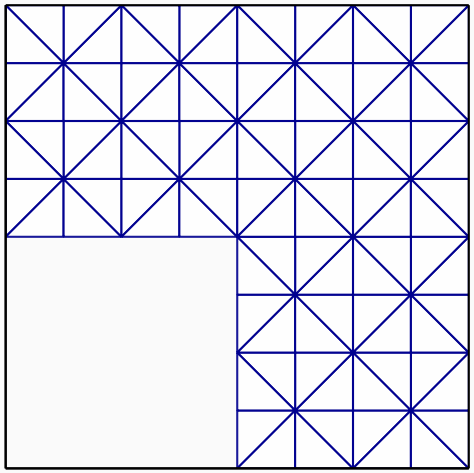
$$\begin{aligned} -\nabla \cdot (\, a(x,\mathbf{y}) \nabla u(x,\mathbf{y})\,) &= 1 && x = (x_1,x_2) \in D, \quad \mathbf{y} \in \Gamma, \\ u(x,\mathbf{y}) &= 0 && x = (x_1,x_2) \in \partial D, \ \mathbf{y} \in \Gamma, \end{aligned}$$

with $a(x,\mathbf{y})$ as given in previous Example 1.

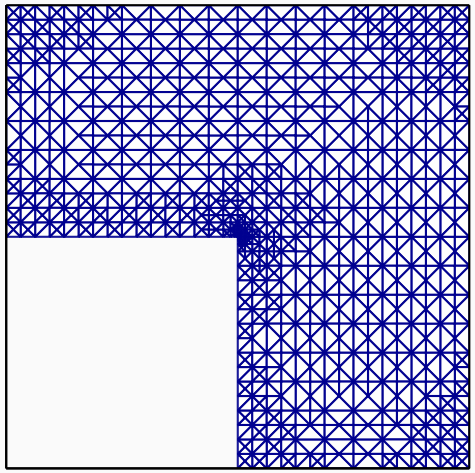
Initial index set: $\mathcal{P}_0 := \{(0,0,0,\dots), (\mathbf{1},0,0,\dots)\}$;

Example 2 (L-shaped domain)

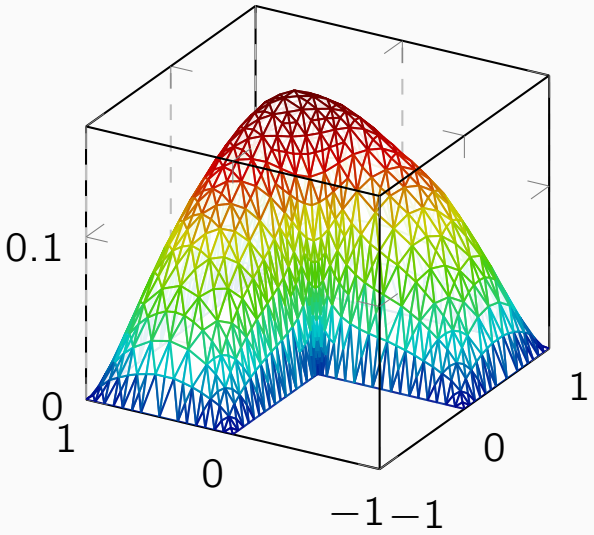
$\#\mathcal{T}_0 = 96$



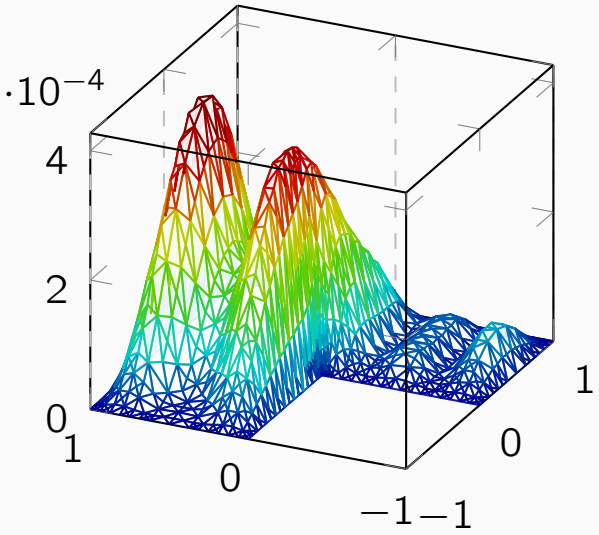
$\#\mathcal{T}_{28} = 1620$



$\mathbb{E}[u_{X\mathcal{P}}]$



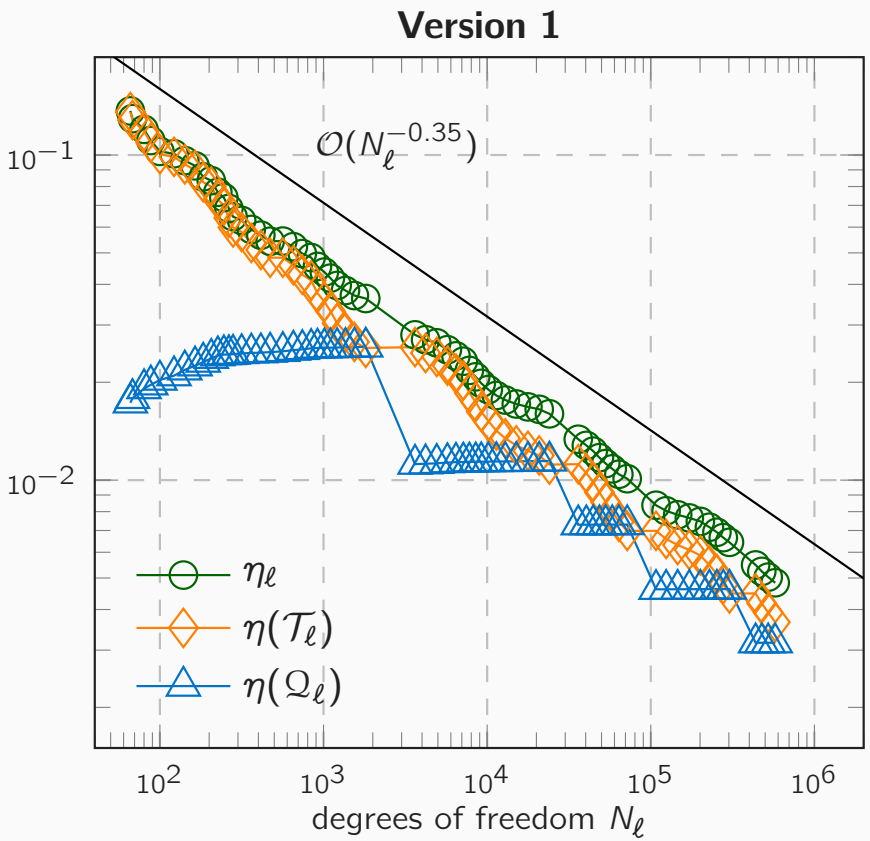
$\mathrm{Var}(u_{X\mathcal{P}})$

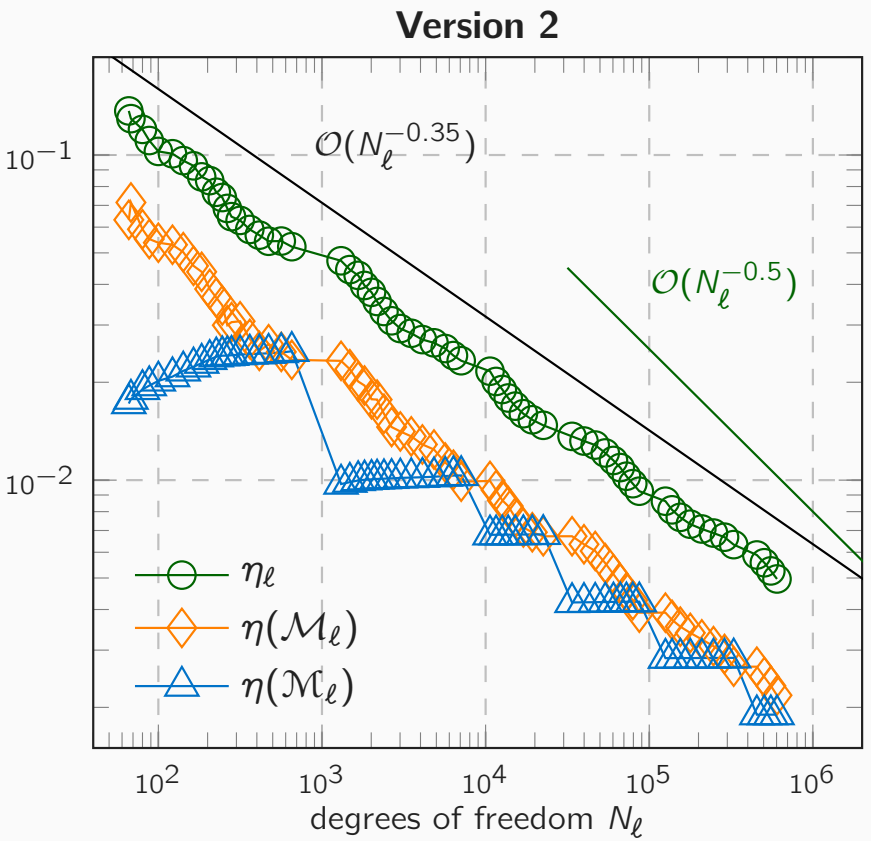


Example 2 (L-shaped domain)

- $\theta_{\mathcal{X}} = 0.2$, $\theta_{\mathcal{P}} = 0.8$, and $\mathrm{TOL} = 5 \cdot 10^{-3}$;

	time (sec)	#iterations	#elements	#indices	#dofs
Version 1	476	65	89,480	13	**576,121**
Version 2	**419**	**63**	**67,770**	**18**	603,594



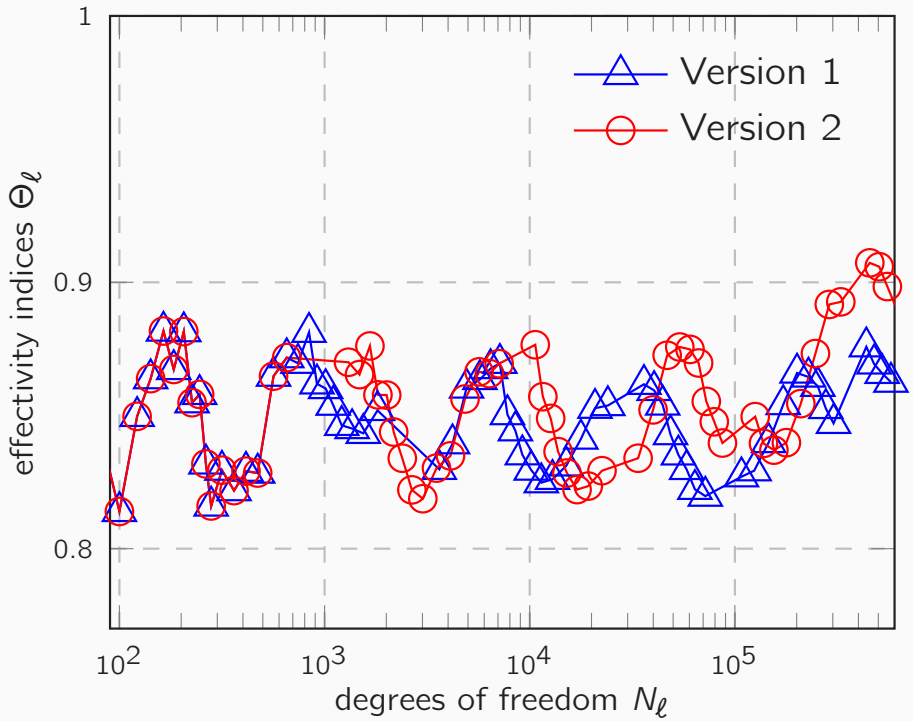


Example 2 (L-shaped domain)

Version 1		Version 2	
$\ell = 0$	$\begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}$	$\ell = 0$
$\ell = 29$	$\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{2} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{2} & 0 \end{pmatrix}$	$\ell = 20$
$\ell = 45$	$\begin{pmatrix} 0 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 \end{pmatrix}$	$\ell = 35$
$\ell = 53$	$\begin{pmatrix} 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 \\ \mathbf{3} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 \\ \mathbf{3} & 0 & 0 & 0 \end{pmatrix}$	$\ell = 43$
$\ell = 62$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{2} & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{2} & \mathbf{1} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{2} & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{2} & \mathbf{1} & 0 & 0 & 0 \end{pmatrix}$	$\ell = 52$
		$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{2} & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{4} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\ell = 60$
#parameters $= 5$	degree $= 3$	#parameters $= 6$	degree $= 4$

Efficiency of the error-estimation: reference solution u_{ref} computed with P_2 approximations over a fine mesh and a large index set ($\sim 27 \cdot 10^6$ dofs):

$$\Theta_\ell := \frac{\eta_\ell}{|||\, u_{\text{ref}} - u_\ell \,|||} = \frac{\eta_\ell}{(|||\, u_{\text{ref}} \,|||^2 - |||\, u_\ell \,|||^2)^{1/2}}$$



Concluding remarks

- Two versions (**marking criterions**) of the adaptive algorithm, based on the comparison of separate error reduction estimates in both spatial and parametric approximations;
- **Optimal rates** during mesh refinement steps provided appropriate marking parameters;
- Efficient and convergent (numerically!) algorithm for parametric problems with both regular and singular solutions.

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Related work (extension)

We use a novel **two-level** error estimate in space $\mu_{Y\mathcal{P}}$ to replace $\|e_{Y\mathcal{P}}\|_0$:

- $\mu_{Y\mathcal{P}}$ does not the require the solution of any residual problem (**the code speeds up!**);
- $\mu_{X\mathcal{P}} := \mu_{Y\mathcal{P}} + \|e_{X\mathcal{Q}}\|_0$ is reliable and efficient.

Concluding remarks

- Two versions (**marking criterions**) of the adaptive algorithm, based on the comparison of separate error reduction estimates in both spatial and parametric approximations;
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Related work (extension)

We use a novel **two-level** error estimate in space $\mu_{\mathcal{YP}}$ to replace $\|e_{\mathcal{YP}}\|_0$:

- $\mu_{\mathcal{YP}}$ does not the require the solution of any residual problem (**the code speeds up!**);
- $\mu_{\mathcal{XP}} := \mu_{\mathcal{YP}} + \|e_{\mathcal{XQ}}\|_0$ is reliable and efficient.
- Application to the **Goal-Oriented** error estimation of Quantities of Interest $G(u)$,

$$G(u) = \mathbb{E}\left[\int_D u(x,\mathbf{y})dx\right], \quad G(u) = \mathbb{E}\left[\int_{T\subset D} \frac{\partial u}{\partial x_1}(x,\mathbf{y})dx\right], \quad G(u) = \mathbb{E}\left[u(x_0,\mathbf{y})\right]\dots$$

⋆ **A. Bespalov, D. Praetorius, L. R., M. Ruggeri**, *Goal-Oriented error estimation and adaptivity for elliptic PDEs with parametric or uncertain inputs*, arXiv:1806.03928, 2018;

Some (main) references:

- **C. Schwab and C. J. Gittelson**, *Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs*, Acta Numer., 20 (2011), pp. 291–467;
- **M. Eigel, C. J. Gittelson, C. Schwab, and E. Zander**, *A convergent adaptive stochastic Galerkin finite element method with quasi-optimal spatial meshes*, ESAIM Math. Model. Numer. Anal., 49 (2015), pp. 1367–1398;
- **A. Bespalov and D. Silvester**, *Efficient adaptive stochastic Galerkin methods for parametric operator equations*, SIAM J. Sci. Comput., 38 (2016), pp. A2118–A2140;
- **A. Bespalov and L. R.**, *Efficient adaptive algorithms for elliptic PDEs with random data*, SIAM/ASA J. Uncertain. Quantif., 6 (2018), pp. 243-272.
- **A. Bespalov and L. R.**, *Stochastic T-IFISS*, January 2018. Available on-line at http://web.mat.bham.ac.uk/A.Bespalov/software/index.html#stoch_tifiss.
- ⋆ **A. Bespalov, D. Praetorius, L. R., M. Ruggeri**, *Goal-Oriented error estimation and adaptivity for elliptic PDEs with parametric or uncertain inputs*, arXiv:1806.03928, 2018;

THANK YOU FOR LISTENING